

Finding a Shortest Vector and More in $2^{n/2+o(n)}$ Time using q -ary Coset Difference Tree

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Abstract

This paper presents a new randomized algorithm for solving the exact shortest vector problem. For the n -dimensional lattice $\mathcal{L}$, our algorithm runs in time and space $2^{n/2+o(n)}$.

Our algorithm can be viewed as a q -ary analogue of the midpoint Hessian for an odd prime q ; more precisely, we use the fact that, for a shortest vector v , the gradient (rather than Hessian) of the periodic Gaussian function at v/q is nearly proportional to v (up to sign), even after aggregation over a relatively large random affine coset. We compute the relevant coset gradient along a chain of intermediate lattices using a combinatorial procedure inspired by Wagner’s generalized birthday algorithm, yielding the $2^{n/2+o(n)}$ time and space complexity.

A variant of the algorithm solves the exact closest vector problem on every input $(y, \mathcal{L})$ with a distance guarantee $\text{dist}(y, \mathcal{L}) \leq 1.039\lambda_1(\mathcal{L})$ within the same time and space complexity. This guarantee holds for a random target and a random lattice drawn according to the Haar-Siegel measure. Thus, this algorithm solves a closest vector problem on such random instances in time and space $2^{n/2+o(n)}$.

Author note & AI use disclosure. The algorithm presented in this paper was suggested by ChatGPT using GPT-5.6 Sol in Ultra mode, building on a number of unsuccessful approaches previously explored by the author. All of the technical analyses were carried out by ChatGPT using GPT-5.6 Sol and verified by the author. The manuscript was written mostly by the author, based on the initial draft generated by AI. The author takes full responsibility for the content; any remaining errors are probably due to human error on the author’s part.

The author is pleased that GPT-5.6 Sol’s astonishing capabilities made it possible to obtain the present results through a combinatorial approach. This is particularly gratifying to him, since he had been exploring such approaches to lattice problems, including those based on generalized birthday and k -SUM algorithms, since reading [8] as a graduate student.

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1 Introduction

An n -dimensional full-rank lattice $\mathcal{L} = \mathcal{L}(B)$ is the set of all integer combinations of the columns of a matrix $B \in \mathbb{R}^{n \times n}$. The shortest vector problem, SVP, asks to find a shortest nonzero vector of $\mathcal{L}$, and the closest vector problem, CVP, asks for a lattice vector closest to a given target. These are central problems in the geometry of numbers, computational complexity, and the concrete security of lattice-based cryptography.

The study of computational lattice problems is shaped by a discrepancy between efficient coarse approximation algorithms and hardness of exact or tight approximations. Polynomial-time lattice-reduction algorithms give useful exponential approximations to SVP and CVP and have led to applications in polynomial factorization, Diophantine approximation, integer programming, and cryptanalysis [38, 39, 31, 21, 48, 37]. In contrast, exact and sufficiently accurate versions of SVP and CVP exhibit strong NP-hardness [11, 40, 32, 26], and serve as security foundation of lattice-based cryptography. The complexity to solve these lattice problems is therefore a central benchmark for lattice algorithms, resulting in the ceaseless improvements in worst-case lattice algorithms [31, 25, 12, 47, 42, 3, 4, 2, 20, 8]. Very recently, $2^{0.5596n+o(n)}$, $2^{0.6039n+o(n)}$ and $2^{0.7314n+o(n)}$ time worst-case SVP algorithms were suggested [33, 24, 28, 23], breaking 2^n time barrier.

Our main results are improved randomized algorithms for these lattice problems.

Theorem 1.1 (Informal). *There are randomized classical algorithms with success probability at least $2/3$ and worst-case time and space $2^{n/2+o(n)}$ for the following problems:*

1. SVP on arbitrary lattices;
2. CVP on targets y satisfying $\text{dist}(y, \mathcal{L}) \leq \alpha \lambda_1(\mathcal{L})$ for $\alpha < 1.039 \dots = \frac{1}{2\sqrt{t_0}}$ where $t_0 = 0.23147 \dots$.

Reference	Kan87/HS07	AKS01	PS09	MV10	ADRS15	CCL18	ACKS25	GFH26a/Kim26	Hhan26	GFH26b	This work
Time	$n^{1/2e}$	$2^{O(n)}$	$2^{2.465n}$	2^{2n}	2^n	$2^{2.05n}$	$2^{1.669n}$	$2^{0.7314n}$	$2^{0.6039n}$	$2^{0.5596n}$	$2^{n/2}$
Space	$\text{poly}(n)$	$2^{O(n)}$	$2^{1.233n}$	2^n	2^n	$2^{n/2}$	$2^{n/2}$	$2^{n/2}$	$2^{n/2}$	$2^{n/2}$	$2^{n/2}$

Classical algorithms for the shortest vector problem. $o(n)$ factors in exponents are omitted.

The first result improves the best known provable complexity for SVP problem of the recent best approaches [33, 24, 28, 23] building on the various SVP algorithms mentioned above. It also solves constant-factor approximate SVP in time $2^{n/2+o(n)}$, which in turn gives $\tilde{O}(n^c)$ -SVP algorithm in time $2^{n/(2c+2)+o(n)}$ using [7, Theorem 5.3]. We emphasize that heuristic algorithms discussed below have better complexity than ours, and our algorithm does not have any practical implication in lattice-based cryptography.

The second result improves the distance guarantee achievable for CVP in time $2^{n/2+o(n)}$, where the previous best was $\text{dist}(y, \mathcal{L}) < 0.422\lambda_1(\mathcal{L})$ [22, 3]. The dimension-preserving reduction of [22, Theorem 6.1] with our CVP algorithm gives a randomized algorithm for 3.67312-CVP in worst-case time and space $2^{n/2+o(n)}$. This gives a randomized $2^{n/2+o(n)}$ time and space algorithm for the approximate shortest independent vector problem and the successive minima problem with the same approximate factor 3.67312 following [41].

We also find an application to random CVP (Corollary 5.9); for a Haar-random lattice $\mathcal{L}$ due to Siegel [49, 46] and random target vector y , we show that $\text{dist}(y, \mathcal{L})/\lambda_1(\mathcal{L}) \leq 1 + O(n^{-1/3})$ holds with overwhelming probability, meaning that we can solve CVP for random target in random lattice in time and space $2^{n/2+o(n)}$.

More related work. Heuristic lattice sieving gives the best known asymptotic estimates for exact SVP and underlies concrete security estimates of lattice-based cryptography [44, 43, 35, 16, 14, 27]. The best heuristic complexities for SVP and CVP are both about $2^{0.2925n+o(n)}$ classically [15, 36]. Heuristic quantum algorithms for SVP have been studied extensively, and the best complexity is $2^{0.2563n+o(n)}$ [13, 34, 19, 18].

Recent work has also strengthened deterministic hardness. An OpenAI technical report gave deterministic $n^{1/400}$ -factor hardness for CVP [45]. Wan proved deterministic hardness of Euclidean SVP within every constant factor [50]. Fine-grained hardness of both problems are also well-studied.

Fine-grained hardness results are well-studied for SVP and CVP and approximate versions [17, 9, 1, 5], and several barriers to strengthening them have been established [1, 6, 29]. No explicit constant lower bound in the exponent is known for both SVP and CVP in the Euclidean norm.

Acknowledgment. The author is grateful to Thijs Laarhoven for discussion about random lattices.

1.1 Technical overview

We present a high-level idea of our algorithm, which is inspired by the recent midpoint Hessian algorithm [28], where the main idea is that, for a shortest vector v , the Hessian of the periodic Gaussian at $v/2$ reveals the direction of v . We will use the gradient of the periodic Gaussian instead, which has a clearer interpretation in our case. In both cases, knowing the direction of v up to an inverse polynomial error suffices to find v itself in time $2^{o(n)}$ using the preprocessing bounded distance decoding (BDD) algorithm from [2] at the preprocessing cost of $2^{0.5n+o(n)}$.

The starting point of our algorithm is similar, that for a shortest vector v , the gradient of the periodic Gaussian at v/q for some odd prime q is almost proportional to v .¹ This choice increases the search space from 2^n midpoints to q^n q -ary points, but also introduces a room for combinatorial optimizations using several cosets as described below. We discuss the barriers to go below $2^{0.5n}$ based on our algorithm and why this approach does not improve the midpoint Hessian algorithm [28] in [Remarks 1.2](#) and [1.3](#).

We focus on the algorithmic structure and the complexity in the overview. The rigorous analysis of the several bounds on estimations will be postponed to the main body of this paper.

Discrete and periodic Gaussians. Let $\mathcal{L}$ be a full-rank lattice of dimension n , and let B be its basis. The dual lattice of $\mathcal{L}$ is defined by $\mathcal{L}^* = \{y : \langle y, x \rangle \in \mathbb{Z} \text{ for every } x \in \mathcal{L}\}$, whose basis is B^{-T} .

For the width parameter $s > 0$, the discrete Gaussian is defined by

$$D_{\mathcal{L},s}(x) := \frac{\rho_s(x)}{\rho_s(\mathcal{L})} \quad (x \in \mathcal{L}) \quad \text{where } \rho_s(x) := \exp\left(-\pi \|x\|^2 / s^2\right). \quad (1)$$

The periodic Gaussian of $\mathcal{L}$ is a $\mathcal{L}$ -periodic function defined by

$$F_{1/s}(z) := \frac{\rho_{1/s}(\mathcal{L} + z)}{\rho_{1/s}(\mathcal{L})}$$

which satisfies $F_{1/s}(z) = F_{1/s}(w)$ if $z - w \in \mathcal{L}$. Let q be an odd prime and identify $q^{-1}\mathcal{L}/\mathcal{L}$ with $\mathbb{F}_q^n$ using a basis B of $\mathcal{L}$, or more formally by the map $w \bmod q \mapsto \frac{Bw}{q} + \mathcal{L}$. The dual basis B^{-T} similarly identifies $\mathcal{L}^*/q\mathcal{L}^*$ with $\mathbb{F}_q^n$ via $z \bmod q \mapsto B^{-T}z + q\mathcal{L}^*$. This gives the following formula, which we will use extensively:

$$\exp\left(2\pi i \left\langle \frac{Bw}{q}, B^{-T}z \right\rangle\right) = \omega_q^{\langle w, z \rangle}.$$

For $w \in \mathbb{F}_q^n$, we define the q -ary periodic Gaussian and its gradient, which will be our main tools, by

$$f(w) := F_{1/s}\left(\frac{Bw}{q}\right), \quad G(w) := \nabla_z F_{1/s}(z)|_{z=Bw/q} = \nabla F_{1/s}\left(\frac{Bw}{q}\right),$$

where any integral lift of w may be used.

We use two different representations of the q -ary periodic Gaussian function and its gradient. Expanding the definition of $F_{1/s}$ and differentiating gives the primal representations:

$$\begin{aligned} f(w) &= \frac{1}{\rho_{1/s}(\mathcal{L})} \sum_{x \in \mathcal{L}} \rho_{1/s}\left(x + \frac{Bw}{q}\right), \\ G(w) &= -\frac{2\pi s^2}{\rho_{1/s}(\mathcal{L})} \sum_{x \in \mathcal{L}} \left(x + \frac{Bw}{q}\right) \rho_{1/s}\left(x + \frac{Bw}{q}\right). \end{aligned}$$

¹For more details, we believe our algorithm should also work with the Hessian instead of gradient, but with a more complicated analysis. On the other hand, for the midpoint $v/2$ as in [28], the gradient vanishes so the choice of Hessian is unavoidable.

This will be used to observe the geometric behavior of these values. On the other hand, Poisson summation formula gives the dual representations, which will be used in the estimations using discrete Gaussian samples obtained from the sampler in [3]:

$$f(w) = \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[\omega_q^{\langle [X]_q, w \rangle} \right], \quad G(w) = 2\pi i \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[X \omega_q^{\langle [X]_q, w \rangle} \right],$$

where $[X]_q := B^T X \bmod q$ denotes the representations of $X \in \mathcal{L}^*/q\mathcal{L}^*$ in $\mathbb{F}_q^n$ and $\omega_q := \exp(2\pi i/q)$.

A gradient pointing toward a shortest vector. Let v be a shortest nonzero vector of $\mathcal{L}$ and let

$$w_v := B^{-1}v \bmod q$$

be the representation of v in $\mathbb{F}_q^n$, so that $Bw_v/q = v/q$ modulo $\mathcal{L}$. For an appropriately chosen width, the term v/q dominates the primal gradient, giving

$$G(w_v) = -\frac{2\pi s^2}{\rho_{1/s}(\mathcal{L})} \sum_{x \in \mathcal{L}} \left(x + \frac{v}{q} \right) \rho_{1/s} \left(x + \frac{v}{q} \right) = -\frac{2\pi s^2}{q} \frac{\rho_{1/s}(v/q)}{\rho_{1/s}(\mathcal{L})} (v + e), \quad \|e\| \leq 2^{-\Omega(n)} \|v\|,$$

where v comes from the term $x = 0$, while e collects the remaining terms. For every $x \neq 0$,

$$\frac{\rho_{1/s}(x + v/q)}{\rho_{1/s}(v/q)} = \exp \left(-\pi s^2 \left(\|x + v/q\|^2 - \|v/q\|^2 \right) \right) \leq \exp(-\Omega(n)),$$

and a standard Gaussian-mass bound shows that their summation satisfies $\|e\| \leq 2^{-\Omega(n)} \|v\|$. Thus $G(w_v)$ points approximately in the direction of v up to the minus sign. Thus, if w_v were known, the gradient at w_v would reveal the direction of v .

The difficulty is that w_v is one of q^n possible points of $q^{-1}\mathcal{L}/\mathcal{L}$. Evaluating the gradient separately at all of them would be much too expensive. The same problem appeared in [28] when the complexity going below 2^n , where they suggested to use the cosets to optimize.

Working with one coset. Let $H \leq \mathbb{F}_q^n$ be a subspace and let $\mathcal{L}_H := \mathcal{L} + \frac{1}{q}BH$ be the corresponding intermediate lattice between $\mathcal{L}$ and $q^{-1}\mathcal{L}$. Instead of treating the points of $\mathbb{F}_q^n$ separately, for each coset $\kappa + H \in \mathbb{F}_q^n/H$ consider the aggregate gradient

$$G(\kappa + H) := \sum_{w \in \kappa + H} G(w) = -\frac{2\pi s^2}{\rho_{1/s}(\mathcal{L})} \sum_{z \in \mathcal{L}_H + B\kappa/q} z \rho_{1/s}(z).$$

Thus summing the q -ary gradients over $\kappa + H$, which we call the coset gradient, can be understood as a (scalar multiple of) the periodic Gaussian gradient over the corresponding coset of the intermediate lattice $\mathcal{L}_H$. In particular, if $w_v \in \kappa + H$, then $G(w_v)$ appears as one of the terms in $G(\kappa + H)$ and points toward v up to minus sign. The coset size will be chosen so that this term remains detectable in the sum, as analyzed later.

On the dual side, by the character orthogonality identity $\sum_{u \in H} \omega_q^{\langle u, y \rangle} = |H| \mathbf{1}_{\{y \in H^\perp\}}$, we have

$$\begin{aligned} G(\kappa + H) &= 2\pi i \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[X \omega_q^{\langle [X]_q, \kappa \rangle} \sum_{u \in H} \omega_q^{\langle [X]_q, u \rangle} \right] \\ &= 2\pi i |H| \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[X \omega_q^{\langle [X]_q, \kappa \rangle} \mathbf{1}_{\{[X]_q \in H^\perp\}} \right]. \end{aligned}$$

Hence summing over a primal coset restricts the dual residue to $H^\perp$. The approach of [28] uses the near-uniform distribution of discrete Gaussian samples among the residue classes to obtain samples conditioned on this event. Since its density is approximately $|H^\perp|/q^n = 1/|H|$, this incurs a multiplicative overhead $|H|$.

Meet-in-the-middle. We instead multiply the gradient by f before taking the coset sum. By independence and $f(-w) = f(w)$, we can write $G(w)f(w)$ using two discrete Gaussian samples as follows

$$\begin{aligned} G(w)f(w) &= G(w)f(-w) = \mathbb{E}_{X_0 \sim D_{\mathcal{L}^*,s}} \left[2\pi i X_0 \omega_q^{\langle w, [X_0]_q \rangle} \right] \mathbb{E}_{X_1 \sim D_{\mathcal{L}^*,s}} \left[\omega_q^{-\langle w, [X_1]_q \rangle} \right] \\ &= \mathbb{E}_{X_0, X_1 \sim D_{\mathcal{L}^*,s}} \left[2\pi i X_0 \omega_q^{\langle w, [X_0]_q - [X_1]_q \rangle} \right]. \end{aligned}$$

Consequently, the sum over the coset $\kappa + H$ becomes

$$\begin{aligned} \sum_{w \in \kappa + H} G(w)f(w) &= \mathbb{E}_{X_0, X_1 \sim D_{\mathcal{L}^*,s}} \left[2\pi i X_0 \omega_q^{\langle \kappa, [X_0]_q - [X_1]_q \rangle} \sum_{u \in H} \omega_q^{\langle u, [X_0]_q - [X_1]_q \rangle} \right] \\ &= |H| \mathbb{E}_{X_0, X_1 \sim D_{\mathcal{L}^*,s}} \left[2\pi i X_0 \omega_q^{\langle \kappa, [X_0]_q - [X_1]_q \rangle} \mathbf{1}_{\{[X_0]_q - [X_1]_q \in H^\perp\}} \right]. \end{aligned} \quad (2)$$

Thus the factor f replaces the condition on one discrete Gaussian sample by a condition on the difference of two ordinary samples.

More generally, a phase-weighted sum over a coset allows this difference to lie in an arbitrary prescribed coset. We explain this phenomenon by considering the coordinate subspaces

$$\mathbb{F}_q^n = H_1 \oplus H_2 \oplus H_3$$

and randomizing this decomposition using a uniform $U \in \text{GL}_n(\mathbb{F}_q)$; here H_2, H_3 are introduced for comparison with the later example. For now, we consider $H_2 \oplus H_3$ together, which parametrizes the cosets of H_1 . Write $Y_r := U[X_r]_q$ for each r . For $\theta \in H_2 \oplus H_3$ and $\delta \in H_1$, we define the phase-weighted version of Eq. (2) by

$$\begin{aligned} A_\delta(\theta) &:= \sum_{\alpha \in H_1} \omega_q^{-\langle \alpha, \delta \rangle} G(U^T(\theta + \alpha)) f(U^T(\theta + \alpha)) \\ &= |H_1| \mathbb{E}_{X_0, X_1 \sim D_{\mathcal{L}^*,s}} \left[2\pi i X_0 \omega_q^{\langle \theta, Y_0 - Y_1 \rangle} \mathbf{1}_{\{(Y_0 - Y_1)_{H_1} = \delta\}} \right]. \end{aligned} \quad (3)$$

The equality follows from $\sum_{\alpha \in H_1} \omega_q^{\langle \alpha, z - \delta \rangle} = |H_1| \mathbf{1}_{\{(z)_{H_1} = \delta\}}$. Thus a phase-weighted sum of Gf over the frequency coset $U^T(\theta + H_1)$ is equivalent to prescribing the H_1 -component of the sample difference. The unweighted coset sum corresponds to $\delta = 0$.

Algorithmically, we estimate $A_\delta(\theta)$ from two independent lists of discrete Gaussian samples by matching pairs satisfying $(Y_0 - Y_1)_{H_1} = \delta$ and averaging their contributions $2\pi i X_0 \omega_q^{\langle \theta, Y_0 - Y_1 \rangle}$. More details will follow.

A four-sample tree. We extend this single difference condition to a tiny tree with three difference conditions. We consider four discrete Gaussian samples $X_0, \dots, X_3$. Let $Y_r := U[X_r]_q$ and define

$$D_{1,0} := Y_0 - Y_1, \quad D_{1,1} := Y_2 - Y_3, \quad D_{2,0} := D_{1,0} - D_{1,1}.$$

Choose independently and uniformly

$$\delta_{1,0}, \delta_{1,1} \in H_1, \quad \delta_{2,0} \in H_2, \quad \boldsymbol{\delta} = (\delta_{1,0}, \delta_{1,1}, \delta_{2,0})$$

and impose the following conditions

$$(D_{1,0})_{H_1} = \delta_{1,0}, \quad (D_{1,1})_{H_1} = \delta_{1,1}, \quad (D_{2,0})_{H_2} = \delta_{2,0}.$$

This gives the quantity analogous to Eq. (3) for $\theta \in H_3$ with the three difference conditions

$$\begin{aligned} A_{\boldsymbol{\delta}}(\theta) &:= |H_1|^2 |H_2| \mathbb{E}_{X_0, \dots, X_3 \sim D_{\mathcal{L}^*,s}} \left[2\pi i X_0 \omega_q^{\langle \theta, D_{2,0} \rangle} \right. \\ &\quad \cdot \mathbf{1}_{\{(D_{1,0})_{H_1} = \delta_{1,0}\}} \mathbf{1}_{\{(D_{1,1})_{H_1} = \delta_{1,1}\}} \mathbf{1}_{\{(D_{2,0})_{H_2} = \delta_{2,0}\}} \left. \right]. \end{aligned}$$

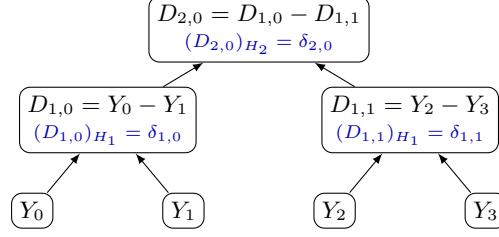


Figure 1: The difference tree for $G(w)f(w)^3$.

Expanding the three conditions by character orthogonality gives the following alternative representation

$$A_{\delta}(\theta) = \sum_{\substack{\alpha_0, \alpha_1 \in H_1 \\ \beta \in H_2}} \omega_q^{-\langle \alpha_0, \delta_{1,0} \rangle + \langle \alpha_1, \delta_{1,1} \rangle - \langle \beta, \delta_{2,0} \rangle} \cdot G(U^T(\theta + \beta + \alpha_0)) f(U^T(\theta + \beta + \alpha_0)) f(U^T(\theta + \beta + \alpha_1))^2. \quad (4)$$

The expansion contains, for a suitable θ , a contribution associated with a shortest vector v . To see this explicitly, recall that $w_v := B^{-1}v \bmod q$ is the $\mathbb{F}_q^n$ representation of v , and let

$$k := U^{-T}w_v = k_1 + k_2 + k_3, \quad k_j := (k)_{H_j}.$$

At $\theta_* := k_3$, the choice $\beta = k_2$ and $\alpha_0 = \alpha_1 = k_1$ contributes

$$c_{\delta}(k)G(w_v)f(w_v)^3, \quad c_{\delta}(k) := \omega_q^{-\langle k_1, \delta_{1,0} \rangle + \langle k_1, \delta_{1,1} \rangle - \langle k_2, \delta_{2,0} \rangle},$$

where $|c_{\delta}(k)| = 1$. This particular term is approximately proportional to v up to a unit complex factor. The components k_1, k_2 are handled implicitly by the sums over $\alpha_0, \alpha_1, \beta$, while k_3 is the only component explicitly indexed by θ , with the desired value $\theta_* = k_3$. The later analysis shows that the remaining terms are sufficiently small.

More concretely, we estimate $A_{\delta}(\theta)$ from four independent lists of discrete Gaussian samples by first matching pairs satisfying the prescribed H_1 -conditions, then matching their differences according to the prescribed H_2 -condition, and summing, with the appropriate normalization, their contributions $2\pi i X_0 \omega_q^{\langle \theta, D_{2,0} \rangle}$.

The general tree. The four-sample construction extends naturally to any $p = 2^{m-1}$ samples. For now, let q be an odd prime and assume for simplicity that $m \mid n$. Decompose

$$\mathbb{F}_q^n = H_1 \oplus \cdots \oplus H_m, \quad |H_1| = \cdots = |H_m| =: Q,$$

where the H_j are coordinate subspaces, and choose a uniform $U \in \text{GL}_n(\mathbb{F}_q)$.

For independent $X_0, \dots, X_{p-1} \sim D_{\mathcal{L}^*, s}$, define

$$D_{0,r} := U[X_r]_q, \quad D_{j,a} := D_{j-1,2a} - D_{j-1,2a+1}$$

for $j = 1, \dots, m-1$ and $a = 0, \dots, p/2^j - 1$. At every node (j, a) , choose an independent uniform $\delta_{j,a} \in H_j$ and impose the difference constraints

$$(D_{j,a})_{H_j} = \delta_{j,a}.$$

Thus, at level j , pairs of elements are matched according to their H_j -components and replaced by their difference before being passed to the next level.

For $\theta \in H_m$, define the corresponding ideal quantity satisfying all the constraints $\delta = (\delta_{j,a})_{j,a}$

$$A_{\delta}(\theta) := Q^{p-1} \mathbb{E}_{X_0, \dots, X_{p-1} \sim D_{\mathcal{L}^*, s}} \left[2\pi i X_0 \omega_q^{\langle \theta, D_{m-1,0} \rangle} \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j - 1} \mathbf{1}_{\{(D_{j,a})_{H_j} = \delta_{j,a}\}} \right]. \quad (5)$$

The normalization is Q^{p-1} because the binary tree has $p-1 = \sum_{j=1}^{m-1} \frac{p}{2^j}$ internal nodes.

Let $\theta_* := (U^{-T}w_v)_{H_m}$. As in the four-sample example, the expansion of Eq. (5) contains the desired term approximately pointing a direction v

$$c_{\delta}(U^{-T}w_v)G(w_v)f(w_v)^{p-1}, \quad |c_{\delta}(U^{-T}w_v)| = 1.$$

Again, the components of $U^{-T}w_v$ in $H_1, \dots, H_{m-1}$ are handled implicitly by the character sums, while its H_m -component is indexed by θ_* . The remaining terms in the expansion are small with high probability:

$$\|A_{\delta}(\theta_*) - c_{\delta}(U^{-T}w_v)G(w_v)f(w_v)^{p-1}\| \leq \frac{s^2 \|v\|}{q} Q^{-1/2} 2^{o(n)}. \quad (6)$$

We will discuss the size of the main term later in the complexity analysis.

Estimation and parameter choice. We now explain how to estimate the ideal quantity in Eq. (5). Recall that $p = 2^{m-1}$, and let N be a list-size parameter to be chosen below. Draw independent samples

$$X_{r,i} \sim D_{\mathcal{L}^*, s}, \quad r \in [0, p), \quad i \in [N],$$

viewed as p lists of size N . For $\mathbf{i} = (i_0, \dots, i_{p-1}) \in [N]^p$, define $D_{j,a}(\mathbf{i})$ recursively from

$$D_{0,r}(\mathbf{i}) := U[X_{r,i_r}]_q, \quad D_{j,a}(\mathbf{i}) := D_{j-1,2a}(\mathbf{i}) - D_{j-1,2a+1}(\mathbf{i}).$$

We then estimate $A_{\delta}(\theta)$ by

$$\hat{A}_{\delta}(\theta) := \frac{Q^{p-1}}{N^p} \sum_{\mathbf{i} \in [N]^p} 2\pi i X_{0,i_0} \omega_q^{\langle \theta, D_{m-1,0}(\mathbf{i}) \rangle} \cdot \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \mathbf{1}_{\{(D_{j,a}(\mathbf{i}))_{H_j} = \delta_{j,a}\}}. \quad (7)$$

Note that this estimation requires the summation over N^p different $\mathbf{i}$. The gradient difference tree algorithm evaluates this sum without enumerating all N^p tuples.

The computation of Eq. (7) for a fixed $\theta \in H_m$ proceeds as follows:

1. Initialize the p lists with the discrete Gaussian samples. This will be the level 0.
2. For each $j = 1, \dots, m-1$, combine pairs from level $j-1$ to form $D_{j,a} = D_{j-1,2a} - D_{j-1,2a+1}$, collecting only those satisfying

$$(D_{j,a})_{H_j} = \delta_{j,a}.$$

3. Sum

$$2\pi i X_{0,i_0} \omega_q^{\langle \theta, D_{m-1,0}(\mathbf{i}) \rangle}$$

over all tuples $\mathbf{i}$ satisfying every difference condition, and multiply the result by Q^{p-1}/N^p .

By construction, this evaluates $\hat{A}_{\delta}(\theta)$ exactly. We analyze its time and space complexity below. To evaluate all $\theta \in H_m$ simultaneously, define and compute the following values for all $b \in H_m$:

$$V_{\delta}(b) := \sum_{\substack{\mathbf{i} \in [N]^p \\ \text{all difference conditions} \\ (D_{m-1,0}(\mathbf{i}))_{H_m} = b}} 2\pi i X_{0,i_0}.$$

Then, we have the representation

$$\hat{A}_{\delta}(\theta) = \frac{Q^{p-1}}{N^p} \sum_{b \in H_m} V_{\delta}(b) \omega_q^{\langle \theta, b \rangle},$$

so one fast Fourier transform over H_m computes the estimator for every $\theta \in H_m$.

Complexity and parameter choice. Now we discuss the complexity of the above algorithm and the parameter choice for the optimized complexity.

Let L_j be the maximum size of a list at level j , with $L_0 = N$. Since each $\delta_{j,a}$ is uniform in H_j , every fixed pair is collected with probability $1/Q$. Thus, for input lists of size at most L_{j-1} , the expected output size for the next level is at most L_{j-1}^2/Q . Grouping by the relevant level j components in complexity² therefore gives

$$S, T = \tilde{O} \left(pN + \sum_{j=1}^{m-1} \frac{p}{2^j} (L_{j-1} + L_j) + Q \log Q \right),$$

where pN is for the discrete Gaussian sampling (here we assume $N \geq 2^{0.5n}$ for the simple use of the discrete Gaussian sampling algorithm in [3]), and the middle term is about collecting the difference satisfying the conditions at each level, and the last term is for the final fast Fourier transform.

To balance the first and the last terms, we choose $N = Qp^{O(1)}$. We also assume $L_j = L_{j-1}^2/Q$ for a simple exposition, so that

$$L_j = Qp^{O(2^j)} = Qp^{O(p)}.$$

By choosing $p \ll n/\log n$, we have $p^{O(p)} = 2^{o(n)}$ and $L_j = Q \cdot 2^{o(n)}$ for all j . Then the complexity becomes

$$T = (N + Q)2^{o(n)}, \quad S = Q2^{o(n)}.$$

Choosing $N = Qp^{O(1)}$ and recalling $Q = (q^n)^{1/m}$ for $m = \log_2 p + 1$, both bounds are $Q2^{o(n)} = 2^{n/2+o(n)}$. In other words, finding the optimal complexity depends on the parameter q .

The constraint on q is related to the size of the desired term and the estimation error

$$\|G(w_v)f(w_v)^{p-1}\| = \frac{s^2 \|v\|}{q} 2^{-(4tp/q^2 - o(1))n}, \quad (\text{estimation error}) = \frac{s^2 \|v\|}{q} Q^{-1/2} 2^{o(n)} \quad (8)$$

hold with high probability. Here $t > 0$ is a fixed constant determined by the Gaussian width and therefore does not affect the asymptotic exponent in this optimization. The first bound (Lemma 3.4) can be shown by the standard Gaussian mass inequalities, and the second bound (Lemma 4.6), together with the above bound Eq. (6) (i.e., Lemma 4.5), is rather complicated to prove.

Since $Q = q^{n/m}$ and $m = \log_2 p + 1$, the above bounds and Eq. (6) show that the desired term is detectable when

$$\frac{4tp}{q^2} \lesssim \frac{\log_2 q}{2 \log_2 p}.$$

This asymptotically enforces $q \gtrsim \sqrt{p}$ or $\log_2 q \gtrsim 0.5 \log_2 p$, which makes

$$Q = q^{n/m} \approx 2^{(\log_2 q / \log_2 p)n} \gtrsim 2^{0.5n}.$$

We choose the parameters accordingly to have $Q = 2^{0.5n+o(n)}$, which is our optimized complexity.

Remark 1.2 (Going below $2^{0.5n}$ seems difficult). The above optimization reaches the complexity $2^{0.5n+o(n)}$ independently of the requirement $N \geq 2^{0.5n}$ imposed by the known discrete Gaussian sampling algorithm [3], which is also used in the preprocessing for BDD [2]. Thus two separate bottlenecks lead to the same exponent: balancing the coset size against the estimation errors, and generating the required discrete Gaussian samples.

Remark 1.3 (The tree algorithm may not improve the midpoint Hessian). See the first bound in Eq. (8). An analogous bound holds for the midpoint Hessian of [28], where $q = 2$ is fixed. Hence increasing p rapidly decreases the norm of the desired term, whereas the error terms do not decrease at a comparable rate, eventually making the desired term too small to detect. The choice $p = 2$ is already used in the essential single-coset optimization.

²Space can be slightly reduced, which is unimportant in the asymptotic analysis.

High-level idea of the analysis. We briefly describe the main idea behind the most technical part of the analysis. The primal expansion of the ideal quantity $A_{\delta}(\theta_*)$ in Eq. (5) contains the desired vector $G(w_v)f(w_v)^{p-1}$. To bound the total squared contribution of the remaining terms, we expand the primal representation. This introduces vectors $u_{r,0}, u_{r,1} \in Bz_r + q\mathcal{L}$ and Gaussian exponents of the form

$$\sum_{r=0}^{p-1} \left(\|u_{r,0}\|^2 + \|u_{r,1}\|^2 \right).$$

Here z_r are coupled by the constraints of the tree, making a direct application of discrete Gaussian mass bounds difficult. We instead repeatedly apply, following the same binary tree, the average–difference transformation

$$(x_L, x_R) \mapsto (a, d) := \left(\frac{x_L + x_R}{2}, x_L - x_R \right).$$

Writing $d_{h,r}$ for the difference created at node r of level h , the resulting full transformation is

$$\Psi : (u_{r,b})_{\substack{0 \leq r < p \\ b \in \{0,1\}}} \mapsto \left(a_{m-1,0}, (d_{h,r})_{\substack{0 \leq h \leq m-1 \\ 0 \leq r < p/2^h}} \right),$$

where $a_{m-1,0}$ is the final average at the root. At each node, the average is passed to its parent and the difference is retained as a new summation variable. The identity $\|x_L\|^2 + \|x_R\|^2 = 2\|a\|^2 + \frac{1}{2}\|d\|^2$ allows us to rewrite the original exponent as a (weighted) sum of the squared norms of a and d 's.

In this representation, the gradient-tree constraints determine the affine lattice coset containing each variable. Together with the injectivity of Ψ , this coset structure allows us to bound the sum by a product of discrete-Gaussian masses, yielding the desired upper bound. In the full proof, we apply this bound recursively through the levels of the tree to control the contribution of all terms other than $G(w_v)f(w_v)^{p-1}$; see Fig. 2.

The CVP extension. Now we move to the CVP problem with the distance guarantee. Let $y \in \mathbb{R}^n$ be a target, let $v \in \mathcal{L}$ be a closest lattice vector to y , and write $e := v - y$ and $w_v := B^{-1}v \bmod q$. Assume that $\|e\| \leq \alpha\lambda_1(\mathcal{L})$ for $\alpha > 0$ to be optimized. We define the shifted q -ary periodic Gaussian and its gradient by

$$\begin{aligned} f_y(w) &:= F_{1/s} \left(\frac{Bw - y}{q} \right) = \mathbb{E}_{X \sim D_{\mathcal{L}^*,s}} \left[\exp \left(-\frac{2\pi i}{q} \langle X, y \rangle \right) \omega_q^{\langle [X]_q, w \rangle} \right], \\ G_y(w) &:= \nabla_z F_{1/s}(z) \big|_{z=(Bw-y)/q} = 2\pi i \mathbb{E}_{X \sim D_{\mathcal{L}^*,s}} \left[X \exp \left(-\frac{2\pi i}{q} \langle X, y \rangle \right) \omega_q^{\langle [X]_q, w \rangle} \right]. \end{aligned}$$

Since $Bw_v = v \bmod q\mathcal{L}$, the periodicity of $F_{1/s}$ gives

$$G_y(w_v) = \nabla F_{1/s} \left(\frac{e}{q} \right) \approx -\frac{2\pi s^2}{q} \frac{\rho_{1/s}(e/q)}{\rho_{1/s}(\mathcal{L})} e.$$

Thus $-G_y(w_v)$ points toward $e = v - y$.

We apply the same coset gradient tree construction to f_y and G_y , obtaining an estimate analogous to Eq. (7). Given $\|e\| \leq \alpha\lambda_1(\mathcal{L})$,³ the desired term satisfies

$$\|G_y(w_v)f_y(w_v)^{p-1}\| \geq \frac{s^2\|e\|}{q} 2^{-(t\alpha^2 + o(1))n}.$$

The difference between the coset gradient tree estimate and $cG_y(w_v)f_y(w_v)^{p-1}$ for some $c \in \mathbb{C}$ with $|c| = 1$ is bounded at most about $Q^{-1/2} = 2^{-n/4}$ up to small factors as above. Therefore, the desired term is detectable when

$$t\alpha^2 < \frac{1}{4}.$$

The lattice-packing bound used in the analysis requires $t > t_0 = 0.23147\dots$. For $\alpha < 1/2\sqrt{t_0}$, the whole procedure have time and space complexity $2^{n/2+o(n)}$ as in the SVP case, with the distance guarantee $\text{dist}(y, \mathcal{L}) \leq 1.039\dots\lambda_1(\mathcal{L})$.

³Technically, we also require $\|e\|$ not to be too small. Otherwise, the closest vector is found directly by the BDD algorithm.

2 Preliminaries

We write $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ to denote the sets of integers, positive integers, real numbers, and complex numbers. q will always be a prime number. Let $\omega_q = \exp(2\pi i/q)$ be the q -th root of unity. $\mathbb{F}_q$ denotes the field of q elements $\{0, 1, \dots, q-1\}$. We write $[m] = \{1, \dots, m\}$ for $m \in \mathbb{N}$ and $[0, m) := \{0, \dots, m-1\}$. For integers $a \leq b$, we write $[a, b) := \{a, a+1, \dots, b-1\}$. The Euclidean norm is denoted by $\|\cdot\|$. We write $\text{dist}(t, S) := \inf_{x \in S} \|t - x\|$ for a set S and a vector t . The indicator function for an event E is denoted by $\mathbf{1}_E$. All logarithms are natural unless a base is specified.

All polynomial factors in this paper are polynomial in the dimension parameter n and the bit length of the inputs, e.g., the input basis; the latter is assumed to be polynomial in n and is omitted in the paper throughout. We suppress routine finite-precision issues and describe arithmetic over $\mathbb{R}$ and $\mathbb{C}$ as exact; polynomially many bits of precision suffice throughout, with only polynomial overhead and errors absorbed into the stated bounds. The dimension n tends to infinity in the asymptotic equations. The constants in the O and Ω notations, as well as $o(1)$ and $o(n)$ terms, are uniform over parameters unless specified otherwise.

2.1 Lattice and discrete Gaussian

Let $B \in \mathbb{R}^{n \times n}$ be a nonsingular matrix and let $\mathcal{L} = B\mathbb{Z}^n$ be a full-rank lattice generated by B . The first minimum is defined by $\lambda_1(\mathcal{L}) = \min_{0 \neq y \in \mathcal{L}} \|y\|$, which is occasionally written as λ . The dual lattice of $\mathcal{L}$ is defined by $\mathcal{L}^* = B^{-T}\mathbb{Z}^n$. Let $\beta = 2^{0.4014\dots}$ denote the lattice-point constant obtained from the spherical-code bound of [30] and [47, Lemma 3]. These bounds are slightly improved in the recent work [45], but we need not use that improved bound.

For $\sigma > 0$ and a countable set A , write $\rho_\sigma(x) = e^{-\pi\|x\|^2/\sigma^2}$ and $\rho_\sigma(A) = \sum_{x \in A} \rho_\sigma(x)$. The centered discrete Gaussian and the smoothing parameter $\eta_\varepsilon(\mathcal{L})$ for $0 < \varepsilon < 1$ are defined by

$$D_{\mathcal{L}, \sigma}(x) = \rho_\sigma(x) / \rho_\sigma(\mathcal{L}), \quad \rho_{1/\eta_\varepsilon(\mathcal{L})}(\mathcal{L}^* \setminus \{0\}) = \varepsilon.$$

For a full-rank lattice M , $\sigma > 0$, and $z \in \mathbb{R}^n$, Poisson summation formula gives

$$\sum_{x \in M} e^{-\pi\|x\|^2/\sigma^2} e^{2\pi i \langle x, z \rangle} = \frac{\sigma^n}{\det M} \sum_{y \in M^*} e^{-\pi\sigma^2\|y-z\|^2} \implies \rho_\sigma(M) = \frac{\sigma^n}{\det M} \rho_{1/\sigma}(M^*)$$

This formula and the triangle inequality show that a centered coset has maximal Gaussian mass.

Lemma 2.1. *For every full-rank lattice M , shift t , and $\sigma > 0$, $\rho_\sigma(M+t) \leq \rho_\sigma(M)$.*

The following theorem shows that the discrete Gaussian sampling can be efficiently done slightly above the smoothing parameter.

Theorem 2.2 ([3, Theorem 5.11]). *Let $\mathcal{L} \subset \mathbb{R}^n$ be a full-rank lattice, $\sigma > 0$, and $\kappa = \Omega(n)$. For every requested $1 \leq M \leq 2^{n/2}$, there is a classical algorithm DGS that outputs M lattice vectors in time $2^{n/2 + \text{polylog}(\kappa) + o(n)}$ and space $2^{n/2 + o(n)}$. If $\sigma > \sqrt{2}\eta_{1/2}(\mathcal{L})$, its output distribution is $\exp(-\Omega(\kappa))$ -close to the distribution of M independent samples $X_1, \dots, X_M$ from $D_{\mathcal{L}, \sigma}$.*

We consider the following lattice problems. We use the search versions by default.

Definition 2.3. *The SVP problem asks to find a nonzero vector $v \in \mathcal{L}$ with $\|v\| = \lambda_1(\mathcal{L})$ given a basis of a full-rank lattice $\mathcal{L}$.*

Definition 2.4. *The CVP problem on input a basis of $\mathcal{L}$ and a target vector y asks to find a vector $v \in \mathcal{L}$ such that $\|y - v\| = \text{dist}(y, \mathcal{L})$.*

Definition 2.5. *For $0 < \alpha < 1/2$, the α -BDD problem on input a basis of $\mathcal{L}$ and a target vector y asks to find the unique closest lattice vector to y , given the promise $\text{dist}(y, \mathcal{L}) < \alpha\lambda_1(\mathcal{L})$.*

We use a time-truncated version [28] of the preprocessing BDD algorithm from [2].

Theorem 2.6. *There is a randomized classical preprocessing algorithm for the $n^{-1/3}$ -BDD problem such that:*

- *its preprocessing runs in worst-case time and space $2^{n/2+o(n)}$ and succeeds with constant probability. The resulting advice has size $2^{o(n)}$;*
- *given a successful preprocessing output, any instance of the $n^{-1/3}$ -BDD problem can be deterministically solved in time $2^{o(n)}$ and polynomial additional space.*

2.2 Useful lemmas

Lemma 2.7. *Let $x, y \in \mathbb{R}^n$ with $y \neq 0$, and let $0 \leq \varepsilon < 1$. If $\|x - y\| \leq \varepsilon \|y\|$, then $x \neq 0$ and*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\varepsilon}{1-\varepsilon}.$$

Proof. The triangle inequality gives $\|x\| \geq \|y\| - \|x - y\| \geq (1 - \varepsilon) \|y\| > 0$. Then we have

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x - y}{\|x\|} + y \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) \right\| \leq \frac{\|x - y\|}{\|x\|} + \|y\| \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \leq \frac{2\|x - y\|}{\|x\|} \leq \frac{2\varepsilon}{1-\varepsilon}. \quad \square$$

3 Preparation

We use the following parameters in the remainder of this paper. Let $\mathcal{L}$ be a full-rank n -dimensional lattice and let $\lambda = \lambda_1(\mathcal{L})$. For $t_0 := \beta^2/(4e \ln 2) = 0.23147\dots$, fix $t_0 < t < 1/4$. Let $d > 0$. We define

$$g = \frac{1}{2} \log_2(t/t_0) > 0, \quad s^2 = \frac{4nt \ln 2}{\pi d^2}.$$

In most of our analysis, we assume that $\lambda \leq d \leq (1 + 1/n)\lambda$. In the algorithm, d is guessed from a polynomial-size list and the main subroutine is run for every value in the list. Only a value satisfying $\lambda \leq d \leq (1 + 1/n)\lambda$ is used in the correctness analysis; outputs from the other guesses do not affect the time complexity and correctness.

Lemma 3.1 ([28, Corollary 3.2]). *Let $\lambda = \lambda_1(\mathcal{L})$, let $\lambda \leq d \leq (1 + 1/n)\lambda$, and let $t_0 < t < 1/4$. Then,*

$$\rho_{1/s}(\mathcal{L}) - 1 \leq 2^{-(1/2+g-o(1))n}, \quad \rho_{\sqrt{2}/s}(\mathcal{L}) - 1 \leq 2^{-(g-o(1))n}, \quad s > \sqrt{2}\eta_{1/2}(\mathcal{L}^*).$$

Lemma 3.2 ([28, Lemma 2.8]). *For every $c > 0$, there is $C_c > 0$ such that, for every lattice $\mathcal{L}$, $\sigma > 0$,*

$$\mathbb{P}_{X \sim D_{\mathcal{L}, \sigma}}[\|X\| > C_c \sigma \sqrt{n}] \leq 2^{-2cn} \quad \text{and} \quad \mathbb{E}_{X \sim D_{\mathcal{L}, \sigma}} \left[\|X\|^2 \mathbf{1}_{\{\|X\| > C_c \sigma \sqrt{n}\}} \right] \leq \sigma^2 n 2^{-cn}.$$

The following lemma is a slight extension of [28, Lemma 3.1]. The proof is unchanged.

Lemma 3.3. *For every fixed integer $k \geq 0$, uniformly over $\mathcal{L}$ and $\sigma > 0$ satisfying $n\sigma^2/\lambda^2 \leq n^{O(1)}$,*

$$\sum_{x \in \mathcal{L} \setminus \{0\}} \left(1 + \frac{\|x\|}{\lambda_1(\mathcal{L})} \right)^k \rho_\sigma(x) \leq 2^{o(n)} \left(\frac{\beta^2 n \sigma^2}{2\pi e \lambda_1^2(\mathcal{L})} \right)^{n/2}.$$

We define the periodic Gaussian [10, 22] for $\mathcal{L}$ with the width $\sigma > 0$ by

$$F_\sigma(z) := \frac{\rho_\sigma(\mathcal{L} + z)}{\rho_\sigma(\mathcal{L})} = \mathbb{E}_{X \sim D_{\mathcal{L}^*, 1/\sigma}} \left[e^{2\pi i \langle X, z \rangle} \right] \quad (9)$$

where the last equality holds because of the Poisson summation formula. Differentiating two representations, we have the following representations of its gradient:

$$\nabla F_\sigma(z) = -\frac{2\pi}{\sigma^2 \rho_\sigma(\mathcal{L})} \sum_{x \in \mathcal{L}} (x+z) \rho_\sigma(x+z) = 2\pi i \mathbb{E}_{X \sim D_{\mathcal{L}^*, 1/\sigma}} \left[X e^{2\pi i \langle X, z \rangle} \right]. \quad (10)$$

Recall q is a prime number. Let B be the basis of $\mathcal{L}$. For $X \in \mathcal{L}^*$, we use the following notation excessively:

$$[X]_q := B^T X \bmod q \in \mathbb{F}_q^n.$$

This is a residue representation of the coset $X + q\mathcal{L}^* \in \mathcal{L}^*/q\mathcal{L}^* \cong (\mathbb{Z}/q\mathbb{Z})^n = \mathbb{F}_q^n$. Recall $s^2 = \frac{4nt \ln 2}{\pi d^2}$ for $t_0 < t < 1/4$. We define the q -ary periodic Gaussian for $w \in \mathbb{F}_q^n$ and its gradient by

$$f(w) := F_{1/s} \left(\frac{Bw}{q} \right) = \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[\omega_q^{\langle [X]_q, w \rangle} \right], \quad G(w) := \nabla F_{1/s} \left(\frac{Bw}{q} \right) = 2\pi i \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[X \omega_q^{\langle [X]_q, w \rangle} \right] \quad (11)$$

using the second representations of Eqs. (9) and (10). For $v \in \mathcal{L}$ and $w = B^{-1}v \bmod q$, periodicity and the first representations in Eqs. (9) and (10) give

$$f(w) = \frac{1}{\rho_{1/s}(\mathcal{L})} \sum_{x \in \mathcal{L}} \rho_{1/s} \left(x + \frac{v}{q} \right), \quad G(w) = -\frac{2\pi s^2}{\rho_{1/s}(\mathcal{L})} \sum_{x \in \mathcal{L}} \left(x + \frac{v}{q} \right) \rho_{1/s} \left(x + \frac{v}{q} \right). \quad (12)$$

We call these representation of f and G as *primal*, and the second representation in Eq. (11) as *dual*. It is not hard to see that f is even and G is odd from the first representations of Eqs. (9) and (10).

The following lemma shows that the gradient of the periodic Gaussian at v/q has direction close to v .

Lemma 3.4. *Let $t_0 < t < 1/4$ and $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Let q be an odd prime. Let $v \in \mathcal{L}$ be a shortest vector and let $w = B^{-1}v \bmod q$. It holds that*

$$f(w) = 2^{-4tn\lambda^2/(d^2q^2)} \left(1 + O(2^{-\Omega(n)}) \right), \quad \left\| -\frac{G(w)}{f(w)} - \frac{2\pi s^2 v}{q} \right\| \leq 2^{-\Omega(n)} s^2 \lambda.$$

Proof. Recall $s^2 = \frac{4nt \ln 2}{\pi d^2}$. For $x \in \mathcal{L} \setminus \{0, -v\}$, it holds that $\|x + v\| \geq \lambda$, which gives $2\langle x, v \rangle \geq -\|x\|^2$ by squaring both sides. Therefore,

$$\|x + v/q\|^2 - \|v/q\|^2 = \|x\|^2 + \frac{2}{q} \langle x, v \rangle \geq \left(1 - \frac{1}{q} \right) \|x\|^2. \quad (13)$$

By the definition of $\rho_{1/s}$ and Eq. (13), the following holds for $x \in \mathcal{L} \setminus \{0, -v\}$

$$\frac{\rho_{1/s}(x + v/q)}{\rho_{1/s}(v/q)} = e^{-\pi s^2 (\|x + v/q\|^2 - \|v/q\|^2)} \leq e^{-\pi s^2 (1-1/q) \|x\|^2} = \rho_{1/(s\sqrt{1-1/q})}(x) \leq \rho_{\sqrt{2}/s}(x), \quad (14)$$

where the last inequality follows from $q \geq 3$ and the fact that $\rho_\sigma(x)$ is increasing in σ . Lemma 3.3 with $\sigma = \sqrt{2}/s$ and $k = 1$ gives, using $s^2 = \frac{4nt \ln 2}{\pi d^2}$ and $g = \frac{1}{2} \log_2(t/t_0)$,

$$\sum_{x \in \mathcal{L} \setminus \{0\}} \left(1 + \frac{\|x\|}{\lambda} \right) \rho_{\sqrt{2}/s}(x) \leq 2^{o(n)} \left(\frac{\beta^2 n}{\pi e \lambda^2 s^2} \right)^{n/2} = 2^{o(n)} \left(\frac{t_0 d^2}{t \lambda^2} \right)^{n/2} \leq 2^{-(g-o(1))n}. \quad (15)$$

For $x = -v$, we have

$$\frac{\rho_{1/s}(-v + v/q)}{\rho_{1/s}(v/q)} = e^{-\pi s^2 (1-2/q) \lambda^2} \leq e^{-\pi s^2 \lambda^2 / 3} = 2^{-\Omega(n)}$$

using $q \geq 3$. Combining this bound with Eqs. (14) and (15) gives

$$\sum_{x \in \mathcal{L} \setminus \{0\}} \left(1 + \frac{\|x\|}{\lambda}\right) \rho_{1/s}(x + v/q) \leq 2^{-\Omega(n)} \rho_{1/s}(v/q). \quad (16)$$

By Lemma 3.1, $\rho_{1/s}(\mathcal{L}) = 1 + 2^{-\Omega(n)}$. It follows from Eqs. (12) and (16) that

$$f(w) = \rho_{1/s}(v/q) \left(1 + 2^{-\Omega(n)}\right) = 2^{-4tn\lambda^2/(d^2q^2)} \left(1 + 2^{-\Omega(n)}\right).$$

Moreover, $\|x + v/q\| \leq \|x\| + \lambda = \lambda(1 + \|x\|/\lambda)$ for every nonzero $x \in \mathcal{L}$. Hence, the gradient representation in Eq. (12) and the triangle inequality, together with Eq. (16), give

$$\left\| G(w) + \frac{2\pi s^2}{\rho_{1/s}(\mathcal{L})} \frac{v}{q} \rho_{1/s}(v/q) \right\| = \frac{2\pi s^2}{\rho_{1/s}(\mathcal{L})} \left\| \sum_{x \in \mathcal{L} \setminus \{0\}} \left(x + \frac{v}{q}\right) \rho_{1/s}\left(x + \frac{v}{q}\right) \right\| \leq \frac{2\pi s^2 \rho_{1/s}(v/q)}{\rho_{1/s}(\mathcal{L})} 2^{-\Omega(n)} \lambda.$$

Finally, dividing the last inequality by $f(w) = \frac{\rho_{1/s}(v/q)}{\rho_{1/s}(\mathcal{L})} (1 + 2^{-\Omega(n)})$ gives the second bound. $\square$

4 The SVP algorithm

We prove the following result.

Theorem 4.1. *There is a randomized classical algorithm that solves SVP with probability at least $2/3$ in worst-case time and space $2^{n/2+o(n)}$.*

We will use the following parameters in the algorithm. Let p be the largest power of two satisfying $p \leq n/\log^3 n$, let q be the smallest prime larger than $2\sqrt{p}$, and let $m = \log_2 p + 1$. Then,

$$p = \Theta(n/\log^3 n), \quad q = (2 + o(1))\sqrt{p}, \quad \log_2 q = \frac{m+1}{2} + o(1) \quad (17)$$

where the asymptotic size of q is due to the prime number theorem.⁴

4.1 The first estimation target

We first explain that it is enough to approximate $G(w)f(w)^{p-1}$ where $f(w)^{p-1}$ is a scalar, but will be essential in the algorithm. The estimation we will see loses the unit factor c , which should be taken into account.

Lemma 4.2. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$ and $t_0 < t < 1/4$, let $v \in \mathcal{L}$ be a shortest vector, and let $w = B^{-1}v \bmod q$. Suppose that $A \in \mathbb{C}^n$ satisfies for some $|c| = 1$ that*

$$\|A - cG(w)f(w)^{p-1}\| \leq \frac{s^2\lambda}{q} 2^{-n/4+o(n)}, \quad (18)$$

then one of the well-defined⁵ nonzero vectors $\text{Re}A/\|\text{Re}A\|$ and $\text{Im}A/\|\text{Im}A\|$ is within the Euclidean distance $2^{-\Omega(n)}$ of either v/λ or $-v/\lambda$. For the corresponding unit vector u (i.e., one of $\text{Re}A/\|\text{Re}A\|$ or $\text{Im}A/\|\text{Im}A\|$), the following holds for all sufficiently large n :

$$\|\pm du - v\| < 2\lambda/n.$$

⁴For every fixed $\eta > 0$, the prime number theorem gives $\pi((1 + \eta)2\sqrt{p}) - \pi(2\sqrt{p}) = (\eta + o(1)) \frac{2\sqrt{p}}{\log(2\sqrt{p})} \geq 1$ for all sufficiently large p .

⁵That is, one with $\|\text{Re}A\| \neq 0$ or $\|\text{Im}A\| \neq 0$.

Proof. By Lemma 3.4 and the choice of q that $q^2 = (4 + o(1))p$, we have

$$f(w)^p = 2^{-(t-o(1))n}, \quad -G(w)f(w)^{p-1} = \frac{2\pi s^2}{q} f(w)^p (v + e) \quad (19)$$

for some $e \in \mathbb{R}^n$ satisfying $\|e\| \leq 2^{-\Omega(n)}\lambda$. By Eqs. (18) and (19) and the triangle inequality,

$$\left\| A + \frac{2\pi s^2}{q} c f(w)^p v \right\| \leq \|A - cG(w)f(w)^{p-1}\| + \frac{2\pi s^2}{q} f(w)^p \|e\| \leq \frac{s^2 \lambda}{q} f(w)^p 2^{-\Omega(n)}.$$

where we use $t < 1/4$ in Eq. (18) to bound the first term. Taking the real and imaginary parts gives

$$\left\| \operatorname{Re} A + \frac{2\pi s^2}{q} (\operatorname{Rec}) f(w)^p v \right\|, \left\| \operatorname{Im} A + \frac{2\pi s^2}{q} (\operatorname{Im} c) f(w)^p v \right\| \leq \frac{s^2 \lambda}{q} f(w)^p 2^{-\Omega(n)}.$$

At least one of $|\operatorname{Rec}|$ and $|\operatorname{Im} c|$ is at least $1/\sqrt{2}$. If $|\operatorname{Rec}| \geq 1/\sqrt{2}$,

$$\left\| \frac{2\pi s^2}{q} (\operatorname{Rec}) f(w)^p v \right\| \geq \frac{\sqrt{2}\pi s^2 \lambda}{q} f(w)^p \gg \frac{s^2 \lambda}{q} f(w)^p 2^{-\Omega(n)}.$$

Thanks to Lemma 2.7, $\operatorname{Re} A$ is nonzero and

$$\left\| \frac{\operatorname{Re} A}{\|\operatorname{Re} A\|} + \operatorname{sgn}(\operatorname{Rec}) \frac{v}{\lambda} \right\| \leq 2^{-\Omega(n)}.$$

An analogous conclusion holds for $\operatorname{Im} A$ if $|\operatorname{Im} c| \geq 1/\sqrt{2}$. The final conclusion holds because for large n

$$\|\pm du - v\| \leq |d - \lambda| + d \|\pm u - v/\lambda\| \leq \frac{\lambda}{n} + 2^{-\Omega(n)}\lambda < 2\lambda/n. \quad \square$$

4.2 The coordinate decomposition and the estimator

Coordinate decomposition. We first present a formula for $G(w)f(w)^{p-1}$ from independent discrete Gaussian samples $X_0, \dots, X_{p-1} \sim D_{\mathcal{L}^*, s}$. We begin with the observation that for any $\varepsilon_r \in \{-1, 1\}$ with $\varepsilon_0 = 1$,

$$\mathbb{E} \left[2\pi i X_0 \omega_q^{\langle w, \sum_{r=0}^{p-1} \varepsilon_r [X_r]_q \rangle} \right] = \mathbb{E} \left[2\pi i X_0 \omega_q^{\langle w, [X_0]_q \rangle} \right] \prod_{r=1}^{p-1} \mathbb{E} \left[\omega_q^{\langle \varepsilon_r w, [X_r]_q \rangle} \right] = G(w) f(w)^{p-1} \quad (20)$$

where we use $\mathbb{E} \left[\omega_q^{\langle \varepsilon_r w, [X_r]_q \rangle} \right] = f(\varepsilon_r w) = f(w)$ because f is an even function. While this formula works for any ε 's, we henceforth let

$$\varepsilon_r := (-1)^{\operatorname{wt}_2(r)} \quad \text{for every } r \in [0, p),$$

where $\operatorname{wt}_2(r)$ is the Hamming weight of the binary representation of r .

For simplicity, we assume that $m = \log_2 p + 1$ divides n throughout in this paper. This assumption does not affect every asymptotic in this paper much, that is, all the changes are at most $2^{o(n)}$.⁶ For $j \in [m]$, let

$$H_j := \{x = (x_1, \dots, x_n) \in \mathbb{F}_q^n : x_k = 0 \text{ for every } k \notin \{(j-1)n/m + 1, \dots, jn/m\}\}.$$

This gives a decomposition

$$\mathbb{F}_q^n = H_1 \oplus \dots \oplus H_m, \quad Y = \sum_{j=1}^m (Y)_{H_j} \quad \text{for } Y \in \mathbb{F}_q^n$$

⁶For the parameters below, we may choose $Q := |H_1| = \dots = |H_{m-1}| = q^\ell$ for $\ell := \lfloor n/m \rfloor$ and $Q_m := |H_m| = q^{n-(m-1)\ell}$. In particular, $Q, Q_m = 2^{n/2+o(n)}$ for our parameter setting.

where $(Y)_{H_j} \in H_j$ denotes the unique component satisfying $Y = \sum_{j=1}^m (Y)_{H_j}$.

We first define the value that we will approximate. Let U be uniformly random in $\text{GL}_n(\mathbb{F}_q)$. For the independent samples $X_0, \dots, X_{p-1}$ in Eq. (20), let $Y_{0,r} := U[X_r]_q = U(B^T X_r \bmod q)$ for $r \in [0, p)$ and define $Y_{j,a} := Y_{j-1,2a} - Y_{j-1,2a+1}$ for $j \in [m-1]$ and $a \in [0, p/2^j)$. For $r \in [0, p)$, let $\varepsilon_r := (-1)^{\text{wt}_2(r)}$, where $\text{wt}_2(r)$ is the Hamming weight of the binary representation of r . It is not hard to see by induction that

$$Y_{m-1,0} = \sum_{r=0}^{p-1} \varepsilon_r Y_{0,r}, \quad Y_{j,a} = \sum_{b=0}^{2^j-1} \varepsilon_b Y_{0,a2^j+b} = \varepsilon_{a2^j} \sum_{r=a2^j}^{(a+1)2^j-1} \varepsilon_r Y_{0,r}. \quad (21)$$

Note that the indices for $Y_{j,a}$ satisfy $a \in [0, p/2^j)$.

We now give an alternative representation of Eq. (20) for $w = B^{-1}v \bmod q$ for a shortest vector v . Let

$$k_* = U^{-T}w, \quad \theta_* := (k_*)_{H_m}, \quad \text{and} \quad \alpha_{j,a}^* := \varepsilon_{a2^j} (k_*)_{H_j} \quad \text{for } j \in [m-1], a \in [0, p/2^j) \quad (22)$$

and let $\alpha^* := (\alpha_{j,a}^*)_{j,a}$. By $Y_{0,r} = U[X_r]_q$ and $k_* = U^{-T}w$, we have

$$\langle w, [X_r]_q \rangle = \langle U^{-T}w, U[X_r]_q \rangle = \langle k_*, Y_{0,r} \rangle \pmod{q}.$$

Thus the exponent $\langle w, \sum_{r=0}^{p-1} \varepsilon_r [X_r]_q \rangle$ in Eq. (20) is $\sum_{r=0}^{p-1} \varepsilon_r \langle k_*, Y_{0,r} \rangle$.

We decompose this sum according to $\mathbb{F}_q^n = H_1 \oplus \dots \oplus H_m$. From the second identity in Eq. (21), we have

$$\sum_{a=0}^{p/2^j-1} \langle \alpha_{j,a}^*, Y_{j,a} \rangle = \sum_{r=0}^{p-1} \varepsilon_r \langle (k_*)_{H_j}, Y_{0,r} \rangle \quad \text{for every } j \in [m-1].$$

The remaining H_m -component satisfies $\langle \theta_*, (Y_{m-1,0})_{H_m} \rangle = \sum_{r=0}^{p-1} \varepsilon_r \langle (k_*)_{H_m}, Y_{0,r} \rangle$ since $\theta_* = (k_*)_{H_m}$ and $Y_{m-1,0} = \sum_{r=0}^{p-1} \varepsilon_r Y_{0,r}$. Combining these identities gives the alternative expression of the exponents in Eq. (20)

$$\omega_q^{\sum_{r=0}^{p-1} \varepsilon_r \langle w, [X_r]_q \rangle} = \omega_q^{\sum_{r=0}^{p-1} \varepsilon_r \langle k_*, Y_{0,r} \rangle} = \omega_q^{\langle \theta_*, (Y_{m-1,0})_{H_m} \rangle} \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{\langle \alpha_{j,a}^*, Y_{j,a} \rangle}. \quad (23)$$

What to estimate. We define an ideal quantity that contains Eq. (20) as a dominating term. We later approximate all its values efficiently from finite sample lists. Let $Q := |H_j| = q^{n/m} = 2^{n/2+o(n)}$. For every $j \in [m-1]$ and $a \in [0, p/2^j)$, let $\delta_{j,a}$ be independently and uniformly distributed over H_j , and let $\delta = (\delta_{j,a})_{j,a}$. For every j and a , character orthogonality gives

$$Q \mathbf{1}_{\{(Y_{j,a})_{H_j} = \delta_{j,a}\}} = \sum_{\alpha_{j,a} \in H_j} \omega_q^{-\langle \alpha_{j,a}, \delta_{j,a} \rangle} \omega_q^{\langle \alpha_{j,a}, Y_{j,a} \rangle}. \quad (24)$$

Here we used $\langle \alpha_{j,a}, (Y_{j,a})_{H_j} \rangle = \langle \alpha_{j,a}, Y_{j,a} \rangle$ because $\alpha_{j,a} \in H_j$. We define the conjunction of all constraints

$$C_\delta(X_0, \dots, X_{p-1}) := \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \mathbf{1}_{\{(Y_{j,a})_{H_j} = \delta_{j,a}\}}. \quad (25)$$

Multiplying Eq. (24) over all pairs (j, a) gives the following, which includes Eq. (23) as a term,

$$Q^{p-1} C_\delta(X_0, \dots, X_{p-1}) = \sum_{\alpha} \left(\prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{-\langle \alpha_{j,a}, \delta_{j,a} \rangle} \right) \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{\langle \alpha_{j,a}, Y_{j,a} \rangle}, \quad (26)$$

where the sum is over $\alpha = (\alpha_{j,a})_{j,a}$ for all $\alpha_{j,a} \in H_j$. This motivates the following definition. For $\theta \in H_m$, let

$$A_{\delta}(\theta) := Q^{p-1} \mathbb{E} \left[C_{\delta}(X_0, \dots, X_{p-1}) 2\pi i X_0 \omega_q^{\langle \theta, (Y_{m-1,0})_{H_m} \rangle} \mid U, \delta \right]. \quad (27)$$

Here the expectation is over the independent samples $X_0, \dots, X_{p-1} \sim D_{\mathcal{L}^*, s}$, while U and δ are fixed. Note that for A_{δ} with the fixed U, δ , the search space is of size $|H_m| = Q = 2^{n/2+o(n)}$.

We later prove the following lemma, stating that one of $A_{\delta}(\theta)$ gives the direction of a shortest vector v .

Lemma 4.3. *Let $v \in \mathcal{L}$ be a shortest vector, let $\lambda = \|v\|$, let $w = B^{-1}v \bmod q$, let $k_* = U^{-T}w$, and let $\theta_* = (k_*)_{H_m}$. Except with probability $o(1)$ over U and δ , there are $c_{\delta} \in \mathbb{C}$ and $e_{\delta} \in \mathbb{C}^n$ satisfying*

$$A_{\delta}(\theta_*) = c_{\delta} G(w) f(w)^{p-1} + e_{\delta} \quad \text{where} \quad |c_{\delta}| = 1 \quad \text{and} \quad \|e_{\delta}\| \leq \frac{s^2 \lambda}{q} 2^{-n/4 - \Omega(n)}.$$

Proof. Substituting Eq. (26) into Eq. (27) gives

$$A_{\delta}(\theta) = \sum_{\alpha} \left(\prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{-\langle \alpha_{j,a}, \delta_{j,a} \rangle} \right) \mathbb{E} \left[2\pi i X_0 \omega_q^{\langle \theta, Y_{m-1,0} \rangle + \sum_{j=1}^{m-1} \sum_{a=0}^{p/2^j-1} \langle \alpha_{j,a}, Y_{j,a} \rangle} \mid U \right],$$

where we used $\theta \in H_m$ and hence $\langle \theta, (Y_{m-1,0})_{H_m} \rangle = \langle \theta, Y_{m-1,0} \rangle$.

For each α and θ , let $k_{\alpha, \theta, 0}, \dots, k_{\alpha, \theta, p-1} \in \mathbb{F}_q^n$ be the coefficients obtained by expanding the recursive differences $Y_{j,a}$ in terms of $Y_{0,0}, \dots, Y_{0,p-1}$, so that

$$\langle \theta, Y_{m-1,0} \rangle + \sum_{j=1}^{m-1} \sum_{a=0}^{p/2^j-1} \langle \alpha_{j,a}, Y_{j,a} \rangle = \sum_{r=0}^{p-1} \langle k_{\alpha, \theta, r}, Y_{0,r} \rangle. \quad (28)$$

Since $Y_{0,r} = U[X_r]_q$, independence of $X_0, \dots, X_{p-1}$, and the last terms in Eq. (11) for G and f give

$$A_{\delta}(\theta) = \sum_{\alpha} \left(\prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{-\langle \alpha_{j,a}, \delta_{j,a} \rangle} \right) G(U^T k_{\alpha, \theta, 0}) \prod_{r=1}^{p-1} f(U^T k_{\alpha, \theta, r}). \quad (29)$$

Let $k_* = U^{-T}w$, $\theta_* = (k_*)_{H_m}$, and $\alpha_{j,a}^* := \varepsilon_{a2^j}(k_*)_{H_j}$. For $\theta = \theta_*$ and $\alpha = \alpha^*$, Eq. (28) gives $k_{\alpha^*, \theta_*, r} = \varepsilon_r k_*$ so that $U^T k_{\alpha^*, \theta_*, r} = \pm w$ for every $r \in [0, p)$. Therefore, the corresponding term in Eq. (29) is

$$c_{\delta} G(w) f(w)^{p-1}, \quad c_{\delta} := \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{-\langle \alpha_{j,a}^*, \delta_{j,a} \rangle},$$

where $|c_{\delta}| = 1$. Defining e_{δ} as the sum of the remaining terms gives

$$A_{\delta}(\theta_*) = c_{\delta} G(w) f(w)^{p-1} + e_{\delta}.$$

By Lemma 4.4 below with $B_{\alpha} = G(U^T k_{\alpha, \theta_*, 0}) \prod_{r=1}^{p-1} f(U^T k_{\alpha, \theta_*, r})$, we have

$$\mathbb{E}_{\delta} \left[\|e_{\delta}\|^2 \mid U \right] = \sum_{\alpha \neq \alpha^*} \left\| G(U^T k_{\alpha, \theta_*, 0}) \prod_{r=1}^{p-1} f(U^T k_{\alpha, \theta_*, r}) \right\|^2.$$

Bounding this term is the hardest part. We defer this part to Lemma 4.5 below, which gives

$$\mathbb{E}_{\delta} \left[\|e_{\delta}\|^2 \mid U \right] \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-n/2 - \Omega(n)}.$$

The claimed bound on $\|e_{\delta}\|$ now follows from Markov's inequality. $\square$

Lemma 4.4. For $\alpha = (\alpha_{j,a})_{j,a}$ with $\alpha_{j,a} \in H_j$, let $B_\alpha \in \mathbb{C}^n$ and let $\chi_\alpha(\delta) := \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{-\langle \alpha_{j,a}, \delta_{j,a} \rangle}$. Then,

$$\mathbb{E}_\delta \left[\left\| \sum_{\alpha} \chi_\alpha(\delta) B_\alpha \right\|^2 \right] = \sum_{\alpha} \|B_\alpha\|^2. \quad (30)$$

Proof. For $\alpha \neq \alpha'$, there is some (j, a) such that $\alpha_{j,a} \neq \alpha'_{j,a}$. Independence and uniformity of the vectors $\delta_{j,a}$ and character orthogonality give

$$\mathbb{E}_\delta [\chi_\alpha(\delta) \overline{\chi_{\alpha'}(\delta)}] = \prod_{j,a} \mathbb{E}_{\delta_{j,a}} \left[\omega_q^{-\langle \alpha_{j,a} - \alpha'_{j,a}, \delta_{j,a} \rangle} \right] = 0.$$

The same expectation is 1 for $\alpha = \alpha'$. Expanding the squared norm and taking expectation gives Eq. (30). $\square$

We defer the proof of the following lemma to Section 4.4.

Lemma 4.5. Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$, let $v \in \mathcal{L}$ be a shortest vector, let $w = B^{-1}v \bmod q$, let $k_* = U^{-T}w$, and let $\theta_* = (k_*)_{H_m}$. Let α^* be defined as in Eq. (22). For each α , let $k_{\alpha, \theta_*, 0}, \dots, k_{\alpha, \theta_*, p-1}$ be defined by Eq. (28) with $\theta = \theta_*$. Except with probability $o(1)$ over U ,

$$\sum_{\alpha \neq \alpha^*} \left\| G(U^T k_{\alpha, \theta_*, 0}) \prod_{r=1}^{p-1} f(U^T k_{\alpha, \theta_*, r}) \right\|^2 \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-n/2 - \Omega(n)}. \quad (31)$$

Estimator. We now define an estimator for A_δ and show that all its values can be computed efficiently. Let $N := Qp^4 = 2^{n/2 + o(n)}$. Recall that U is sampled uniformly from $\text{GL}_n(\mathbb{F}_q)$. We consider N tuples of samples: For every $r \in [0, p)$ and $i \in [N]$, choose independent samples

$$X_{r,i} \sim D_{\mathcal{L}^*, s}, \quad Y_{r,i} := U[X_{r,i}]_q.$$

For $\mathbf{i} = (i_0, \dots, i_{p-1}) \in [N]^p$, we consider the following indexed versions of Y 's

$$Y_{0,r}(\mathbf{i}) := Y_{r,i_r} \text{ for } r \in [0, p), \text{ and } Y_{j,a}(\mathbf{i}) := Y_{j-1,2a}(\mathbf{i}) - Y_{j-1,2a+1}(\mathbf{i}) \text{ for } j \in [m-1], a \in [0, p/2^j]. \quad (32)$$

Note that all vectors $Y_{j,a}(\mathbf{i})$ are determined by the samples X_{r,i_r} for $r \in [0, p)$. Let

$$C_\delta(\mathbf{i}) := C_\delta(X_{0,i_0}, \dots, X_{p-1,i_{p-1}}).$$

For $\theta \in H_m$, define the following estimator for Eq. (27):

$$\widehat{A}_\delta(\theta) := \frac{Q^{p-1}}{N^p} \sum_{\mathbf{i} \in [N]^p} C_\delta(\mathbf{i}) 2\pi i X_{0,i_0} \omega_q^{\langle \theta, (Y_{m-1,0}(\mathbf{i}))_{H_m} \rangle}. \quad (33)$$

We prove the accuracy of the estimator for using Lemma 4.2. The first step is the following lemma, whose proof is deferred to Section 4.4 due to its complexity.

Lemma 4.6. Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $2^{-\Omega(n)}$ over U , simultaneously for every $\theta \in H_m$,

$$\mathbb{E}_{\delta, X} \left[\left\| \widehat{A}_\delta(\theta) - A_\delta(\theta) \right\|^2 \mid U \right] \leq \left(\frac{s^2 \lambda}{q} \right)^2 Q^{-1} 2^{o(n)}.$$

Lemma 4.7. Let $v \in \mathcal{L}$ be a shortest vector, let $\lambda = \|v\|$, let $w = B^{-1}v \bmod q$, and let $\theta_* = (U^{-T}w)_{H_m}$. Except with probability $o(1)$ over U , δ , and the samples $X_{r,i}$, there is $c_\delta \in \mathbb{C}$ with $|c_\delta| = 1$ such that

$$\left\| \widehat{A}_\delta(\theta_*) - c_\delta G(w) f(w)^{p-1} \right\| \leq \frac{s^2 \lambda}{q} 2^{-n/4 + o(n)}. \quad (34)$$

Proof. By Lemma 4.6 and $Q = 2^{n/2+o(n)}$, except with probability $2^{-\Omega(n)}$ over U ,

$$\mathbb{E}_{\delta, X} \left[\left\| \hat{A}_\delta(\theta_*) - A_\delta(\theta_*) \right\|^2 \mid U \right] \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-n/2+o(n)}.$$

Markov's inequality, after choosing the $o(n)$ term sufficiently large, shows that except with probability $o(1)$ over δ and the samples,

$$\left\| \hat{A}_\delta(\theta_*) - A_\delta(\theta_*) \right\| \leq \frac{s^2 \lambda}{q} 2^{-n/4+o(n)}.$$

By Lemma 4.3, except with probability $o(1)$, there is $c_\delta \in \mathbb{C}$ with $|c_\delta| = 1$ such that

$$\left\| A_\delta(\theta_*) - c_\delta G(w) f(w)^{p-1} \right\| \leq \frac{s^2 \lambda}{q} 2^{-n/4-\Omega(n)}.$$

The triangle inequality proves Eq. (34). $\square$

We conclude this section by proving an efficient batch computation of estimator at all θ .

Lemma 4.8. *Except with probability $o(1)$ over δ , all vectors $\hat{A}_\delta(\theta)$ for $\theta \in H_m$ can be computed in time and space $2^{n/2+o(n)}$.*

Algorithm 1: Computation of $\hat{A}_\delta$

1. For every $r \in [0, p)$ and $i \in [N]$, let $Z_{0,i} = 2\pi i X_{0,i}$ and $Z_{r,i} = 0$ for $r \neq 0$. Construct lists

$$\mathbf{L}_{0,r} := \{(i, Y_{r,i}, Z_{r,i}) : i \in [N]\} \quad \text{for } r \in [0, p).$$

2. For $j = 1, \dots, m-1$ and $a \in [0, p/2^j)$, we do:

- (a) For each $z \in H_j$, construct $\mathbf{L}_{j-1,2a+1,z} = \{(i_R, Y_R, Z_R) \in \mathbf{L}_{j-1,2a+1} : (Y_R)_{H_j} = z\}$.
- (b) For every $(i_L, Y_L, Z_L) \in \mathbf{L}_{j-1,2a}$, enumerate all elements (i_R, Y_R, Z_R) in $\mathbf{L}_{j-1,2a+1,(Y_L)_{H_j} - \delta_{j,a}}$ and include $((i_L, i_R), Y_L - Y_R, Z_L + Z_R)$ in $\mathbf{L}_{j,a}$. Abort immediately if $|\mathbf{L}_{j,a}| > Qp^{7 \cdot 2^j}$.

3. For every $b \in H_m$, compute $T_\delta(b) := \sum_{\substack{(i,Y,Z) \in \mathbf{L}_{m-1,0} \\ (Y)_{H_m} = b}} Z$.

4. Using a vector-valued fast Fourier transform over H_m , compute for every $\theta \in H_m$

$$\hat{A}_\delta(\theta) = \frac{Q^{p-1}}{N^p} \sum_{b \in H_m} T_\delta(b) \omega_q^{\langle \theta, b \rangle}. \quad (35)$$

Proof. We call the first index of the list $\mathbf{L}$ by level. We first describe the elements in each list. The list $\mathbf{L}_{j,a}$ is obtained from the initial lists indexed by $a2^j, \dots, (a+1)2^j - 1$. We claim by induction on j that, for every choice of indices $i_{a2^j}, \dots, i_{(a+1)2^j-1} \in [N]$, the algorithm creates at most one element of the form

$$\left((i_{a2^j}, \dots, i_{(a+1)2^j-1}), Y = \sum_{\ell=0}^{2^j-1} \varepsilon_\ell Y_{a2^j+\ell, i_{a2^j+\ell}}, Z \right)$$

in the list $\mathbf{L}_{j,a}$ where $Z = 2\pi i X_{0,i_0}$ if $a = 0$ and $Z = 0$ otherwise. Moreover, a full index tuple $\mathbf{i} = (i_0, \dots, i_{p-1})$ appears in $\mathbf{L}_{m-1,0}$ if and only if $C_\delta(\mathbf{i}) = 1$.

The statement holds for $j = 0$ by the definition of $\mathbf{L}_{0,a}$. Suppose that it holds at level $j - 1$. An element in $\mathbf{L}_{j,a}$ is obtained from the elements in $\mathbf{L}_{j-1,2a}$ and $\mathbf{L}_{j-1,2a+1}$. Its index tuple is the concatenation of the two index tuples, its second component is $Y_L - Y_R$, and its third component is $Z_L + Z_R$. Since

$$\varepsilon_{2^{j-1}+\ell} = -\varepsilon_\ell \quad \text{for } 0 \leq \ell < 2^{j-1},$$

the stated formulas for Y and Z follow. The last claim on $C_\delta(\mathbf{i})$ is not hard to see, because Step 2 combines the elements with Y_R and Y_L if and only if $(Y_L - Y_R)_{H_j} = \delta_{j,a}$, and a tuple appears at level j if and only if all the corresponding conditions up to level j are satisfied. For $j = m - 1$, this means that all the constraints are satisfied, meaning that $C_\delta(\mathbf{i}) = 1$.

Because of this claim, $\mathbf{L}_{m-1,0}$ contains $(\mathbf{i}, Y_{m-1,0}(\mathbf{i}), 2\pi i X_{0,i_0})$ such that $C_\delta(\mathbf{i}) = 1$. Hence Step 3 gives

$$T_\delta(b) = \sum_{\substack{\mathbf{i} \in [N]^p \\ (Y_{m-1,0}(\mathbf{i}))_{H_m} = b}} C_\delta(\mathbf{i}) 2\pi i X_{0,i_0}.$$

Substituting this equality into Eq. (35) gives exactly Eq. (33). This gives the correctness of each $\hat{A}_\delta(\theta)$.

It remains to compute the complexity. We first bound the list sizes. Fix U , all samples, and a candidate index tuple $\mathbf{i}$. Then all vectors $Y_{h,b}(\mathbf{i})$ are fixed. The conditions for the corresponding tuple to appear in $\mathbf{L}_{j,a}$ are $(Y_{h,b})_{H_h} = \delta_{h,b}$ for $h \in [j]$ and $a2^{j-h} \leq b < (a+1)2^{j-h}$. The number of them is $2^j - 1$. Since the vectors $\delta_{h,b}$ are independent and uniform over H_h , each condition holds with probability $1/|H_h| = 1/Q$. Therefore, all the conditions hold with probability $Q^{-(2^j-1)}$. By the linearity of the expectation,

$$\mathbb{E}_\delta[|\mathbf{L}_{j,a}|] = \frac{N^{2^j}}{Q^{2^j-1}} = Qp^{4 \cdot 2^j}.$$

By Markov's inequality, $\Pr_\delta[|\mathbf{L}_{j,a}| > Qp^{7 \cdot 2^j}] \leq p^{-3 \cdot 2^j}$. Because there are $p/2^j$ lists $\mathbf{L}$ at level j ,

$$\Pr_\delta[\text{the algorithm aborts}] \leq \sum_{j=1}^{m-1} \frac{p}{2^j} p^{-3 \cdot 2^j} \leq p^{-5} \sum_{j=1}^{m-1} 2^{-j} \leq p^{-5} = o(1) \quad (36)$$

by a union bound, where we use $2^j \geq 2$ for every $j \geq 1$ and $p \rightarrow \infty$.

Suppose that the algorithm does not abort. The lists $\mathbf{L}_{j-1,2a+1,z}$ in Step 2 are constructed by sorting $\mathbf{L}_{j-1,2a+1}$ according to the H_j -coordinate of its second component, which dominates the second step. Given the size bound $Qp^{7 \cdot 2^j}$, the total number of elements over all levels is at most

$$\sum_{j=0}^{m-1} \frac{p}{2^j} Qp^{7 \cdot 2^j} \leq 2pQp^{7p} \leq Qp^{7p+2} = 2^{n/2+o(n)} \quad (37)$$

because $2^{m-1} = p$ and we use $2^j \leq p$ in the exponent. The final bound is due to $p = \Theta(n/\log^3 n)$. All other steps incur only polynomial multiplicative factors.

Up to polynomial factors, the third step can be done in time and space about $Q + |\mathbf{L}_{m-1,0}| = 2^{n/2+o(n)}$ after classifying the elements in $\mathbf{L}_{m-1,0}$ based on $(Y)_{H_m}$, and the last step is done in time and space about $|H_m| = Q = 2^{n/2+o(n)}$. Together with Eqs. (36) and (37), this proves the lemma. $\square$

4.3 The algorithm

Parameters. Fix a constant t satisfying $t_0 < t < 1/4$, and retain the parameters from Eq. (17): let p be the largest power of two with $p \leq n/\log^3 n$, let $m = \log_2 p + 1$, and let q be the smallest prime larger than $2\sqrt{p}$. Set $N := Qp^4$. Then

$$p \log p = O(n/\log^2 n) = o(n), \quad Q, N = 2^{n/2+o(n)}.$$

Input. A basis $B \in \mathbb{R}^{n \times n}$ of a full-rank lattice $\mathcal{L} = B\mathbb{Z}^n$.

Output. A shortest nonzero vector of $\mathcal{L}$ with probability at least $2/3$.

Algorithm 2: The SVP algorithm

1. Apply LLL reduction to obtain a nonzero $x \in \mathcal{L}$ with $\|x\| \leq 2^{n/2} \lambda_1(\mathcal{L})$, and let

$$\mathcal{D} := \{(1 + 1/n)^{-j} \|x\| : 0 \leq j \leq n^2\}.$$

2. For every $d \in \mathcal{D}$, independently repeat the following procedure a constant number of times:
 - (a) Run the preprocessing algorithm in [Theorem 2.6](#).
 - (b) Let $s^2 = 4nt \ln 2 / (\pi d^2)$ and obtain pN samples $X_{r,1}, \dots, X_{r,N}$ from $D_{\mathcal{L}^*,s}$ for $r \in [0, p)$. Abort this repetition if a sample has norm larger than $Cs\sqrt{n}$, where C is a sufficiently large constant.
 - (c) Choose random U and δ , and run [Algorithm 1](#). Abort this repetition if that algorithm aborts.
 - (d) For every $\theta \in H_m$ and every nonzero vector $V \in \{\text{Re}\hat{A}_\delta(\theta), \text{Im}\hat{A}_\delta(\theta)\}$, query the BDD algorithm at both $dV/\|V\|$ and $-dV/\|V\|$.
 3. Among all nonzero lattice vectors returned by the BDD algorithm, return one of minimum Euclidean norm. If none was returned, report failure.
-

Proof of [Theorem 4.1](#). Fix a shortest vector v , let $\lambda = \|v\|$, and consider $d \in \mathcal{D}$ satisfying $\lambda \leq d \leq (1 + 1/n)\lambda$, whose existence follows from the LLL bound [\[38\]](#) and the definition of $\mathcal{D}$. By [Lemma 3.1](#), the parameter s satisfies the assumption of [Theorem 2.2](#) with (say) $\kappa = n^2$. Hence, the required pN samples are obtained using $2^{o(n)}$ calls, and their joint distribution is inverse-exponentially close to that of independent discrete Gaussian samples.

Let $w = B^{-1}v \bmod q$ and $\theta_* = (U^{-T}w)_{H_m}$. Note that $w \neq 0$, since otherwise $v = qv'$ for some nonzero $v' \in \mathcal{L}$, contradicting the minimality of v . By [Lemmas 4.2](#) and [4.7](#), except with probability $o(1)$, one of the targets queried to the BDD algorithm for $\theta = \theta_*$ is within $2\lambda/n < n^{-1/3}\lambda$ of v . Therefore, successful BDD preprocessing returns v . Since v is among the returned vectors and every nonzero lattice vector has norm at least λ , Step 3 returns a shortest vector.

It remains to bound the complexity. Since $pN = Qp^5 = 2^{n/2+o(n)}$, the DGS calls and [Lemma 4.8](#) use time and space $2^{n/2+o(n)}$. The BDD-query step makes $Q2^{o(n)}$ queries in each repetition, whose total time is also $2^{n/2+o(n)}$. The $n^2 + 1$ choices of d and the constant number of repetitions incur only a polynomial factor.

By [Lemmas 3.2](#), [4.7](#) and [4.8](#) and [Theorem 2.6](#), each repetition for the value d fixed above succeeds with constant probability and has worst-case time and space $2^{n/2+o(n)}$. A sufficiently large constant number of independent repetitions raises the success probability to at least $2/3$. $\square$

4.4 Proofs of the lemmas

[Fig. 2](#) shows a high-level flow of the analysis of the proof of [Lemma 4.5](#).

Gaussian mass bounds. For $j \in \{0, \dots, m-1\}$, let

$$\begin{aligned} P_j &:= H_1 \oplus \dots \oplus H_j, & S_j &:= U^T P_j, & R_j &:= H_{j+1} \oplus \dots \oplus H_m, \\ K_j &:= \{Bx : x \in \mathbb{Z}^n, x \bmod q \in S_j\}, & \sigma_j &:= \frac{q}{2^{j/2} s}, \end{aligned} \tag{38}$$

where $P_0 = S_0 = \{0\}$. Thus $\mathbb{F}_q^n = P_j \oplus R_j$. Thus R_j contains exactly one representative of every coset of P_j , and the map $\kappa + P_j \mapsto U^T \kappa + S_j$ is a bijection from $\mathbb{F}_q^n / P_j$ to $\mathbb{F}_q^n / S_j$. Recall $g = \frac{1}{2} \log_2(t/t_0) > 0$ and $s^2 = 4nt \ln 2 / (\pi d^2)$, where $t_0 < t < 1/4$. Throughout this subsection, t is a fixed constant, and hence $g = \Omega(1)$.

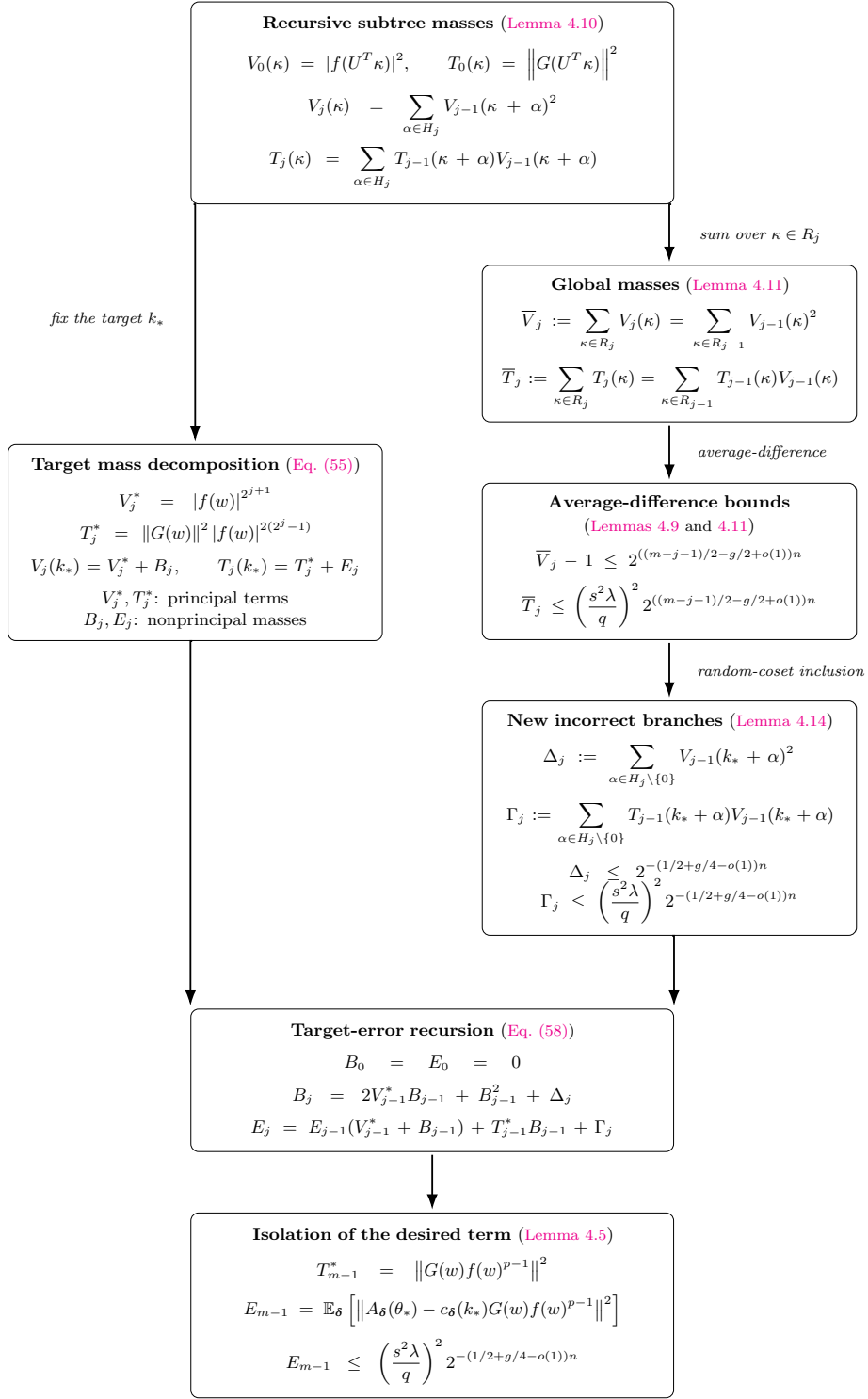


Figure 2: The outline of the proof of the nonprincipal-mass analysis Lemma 4.5. We need a similar but more involved proof for Lemma 4.6.

We use the following Gaussian mass bounds regarding the cosets. Note that the second inequality has a factor $1/\sqrt{1-1/n}$ in the width. Looking ahead, this term is, in the norm of gradient terms, to absorb the scalar factor in the exponent at the cost of small factors; e.g., $xe^{-x} \leq ne^{-(1-1/n)x}$.

Lemma 4.9. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $2^{-\Omega(n)}$ over U , simultaneously for every $j \in \{0, \dots, m-1\}$,*

$$\rho_{\sigma_j}(K_j \setminus \{0\}) \leq 2^{-(g/2-o(1))n}, \quad \rho_{\sigma_j/\sqrt{1-1/n}}(K_j \setminus \{0\}) \leq 2^{-(g/2-o(1))n}. \quad (39)$$

Proof. Since U is uniform in $\text{GL}_n(\mathbb{F}_q)$, the subspace $S_j = U^T P_j$ is uniform among all jn/m -dimensional subspaces of $\mathbb{F}_q^n$. Consequently, every fixed nonzero vector of $\mathbb{F}_q^n$ belongs to S_j with probability

$$\frac{q^{jn/m} - 1}{q^n - 1} \leq 2q^{-(n-jn/m)}.$$

The sublattice $q\mathcal{L}$ is contained in K_j independently of U , and $\rho_\sigma(q\mathcal{L} \setminus \{0\}) = \rho_{\sigma/q}(\mathcal{L} \setminus \{0\})$. Therefore, for every $\sigma > 0$,

$$\begin{aligned} \mathbb{E}_U[\rho_\sigma(K_j \setminus \{0\})] &= \rho_\sigma(q\mathcal{L} \setminus \{0\}) + \sum_{x \in \mathcal{L} \setminus q\mathcal{L}} \Pr_U[x \in K_j] \rho_\sigma(x) \\ &\leq \rho_{\sigma/q}(\mathcal{L} \setminus \{0\}) + 2q^{-(n-jn/m)} \rho_\sigma(\mathcal{L} \setminus \{0\}). \end{aligned} \quad (40)$$

We apply this inequality with $\sigma = \sigma_j$. By Lemma 3.3 with $k = 0$, first at width σ_j/q and then at width σ_j ,

$$\begin{aligned} \rho_{\sigma_j/q}(\mathcal{L} \setminus \{0\}) &\leq 2^{o(n)} \left(\frac{\beta^2 n}{2\pi e \lambda^2 2^j s^2} \right)^{n/2} = 2^{-(g+(j+1)/2-o(1))n}, \\ \rho_{\sigma_j}(\mathcal{L} \setminus \{0\}) &\leq 2^{o(n)} \left(\frac{\beta^2 n q^2}{2\pi e \lambda^2 2^j s^2} \right)^{n/2} = 2^{(\log_2 q - (j+1)/2 - g + o(1))n}. \end{aligned}$$

Here we used $d/\lambda = 1 + O(1/n)$ and $t_0 = \beta^2/(4e \ln 2)$. Since $\log_2 q = (m+1)/2 + o(1)$, substituting these bounds into Eq. (40) gives

$$\mathbb{E}_U[\rho_{\sigma_j}(K_j \setminus \{0\})] \leq 2^{-(g+(j+1)/2-o(1))n} + 2^{-(g+(m-j)/(2m)-o(1))n} \leq 2^{-(g-o(1))n}.$$

Replacing σ_j by $\sigma_j/\sqrt{1-1/n}$ multiplies each upper bound obtained from Lemma 3.3 by at most $(1-1/n)^{-n/2} = 2^{O(1)}$. Hence, the same argument gives

$$\mathbb{E}_U \left[\rho_{\sigma_j/\sqrt{1-1/n}}(K_j \setminus \{0\}) \right] \leq 2^{-(g-o(1))n}.$$

Markov's inequality and a union bound over the $2m$ estimates prove the lemma. $\square$

Recursive mass representations. For $\kappa \in \mathbb{F}_q^n$, we define

$$V_0(\kappa) := |f(U^T \kappa)|^2, \quad T_0(\kappa) := \|G(U^T \kappa)\|^2. \quad (41)$$

For $j \in \{1, \dots, m-1\}$ and $\kappa \in \mathbb{F}_q^n$, we iteratively define

$$V_j(\kappa) := \sum_{\alpha \in H_j} V_{j-1}(\kappa + \alpha)^2, \quad T_j(\kappa) := \sum_{\alpha \in H_j} T_{j-1}(\kappa + \alpha) V_{j-1}(\kappa + \alpha). \quad (42)$$

Induction shows that V_j and T_j are even and P_j -periodic: for every $\kappa \in \mathbb{F}_q^n$ and $\gamma \in P_j$,

$$V_j(-\kappa) = V_j(\kappa), \quad T_j(-\kappa) = T_j(\kappa), \quad V_j(\kappa + \gamma) = V_j(\kappa), \quad T_j(\kappa + \gamma) = T_j(\kappa).$$

Consequently, $V_j(\kappa)$ and $T_j(\kappa)$ depend only on the coset $U^T \kappa + S_j$. Under the bijection $\kappa + P_j \mapsto U^T \kappa + S_j$, the recurrence in Eq. (42) can be seen as the recurrence obtained by summing over the S_{j-1} -cosets contained in a fixed S_j -coset.

We first explain the meaning of V_j and T_j . For $j \in \{0, \dots, m-1\}$, let

$$\mathcal{A}_{\leq j} := \left\{ (\alpha_{h,a})_{\substack{1 \leq h \leq j \\ 0 \leq a < 2^{j-h}}} : \alpha_{h,a} \in H_h \right\}.$$

We write an element of $\mathcal{A}_{\leq j}$ as $\alpha^{\leq j}$. The set $\mathcal{A}_{\leq 0}$ contains only the empty tuple. At $j = m-1$, its indices are $1 \leq h \leq m-1$ and $0 \leq a < p/2^h$, so we identify $\alpha^{\leq m-1}$ with the tuple α used in Eq. (26). For $\kappa \in \mathbb{F}_q^n$, $\alpha^{\leq j} \in \mathcal{A}_{\leq j}$, and $r \in [0, 2^j]$, let

$$k_{\alpha^{\leq j}, \kappa, r} := \varepsilon_r \kappa + \sum_{h=1}^j \varepsilon_{r \bmod 2^h} \alpha_{h, \lfloor r/2^h \rfloor}. \quad (43)$$

We prove the following lemma. At $j = m-1$ and $\kappa = \theta \in H_m$, the term $k_{\alpha^{\leq m-1}, \theta, r}$ in Eq. (43) agree with $k_{\alpha, \theta, r}$ from Eq. (28). Since $2^{m-1} = p$, this gives the following equality (cf. Eq. (31)):

$$T_{m-1}(\theta) = \sum_{\alpha} \left\| G(U^T k_{\alpha, \theta, 0}) \prod_{r=1}^{p-1} f(U^T k_{\alpha, \theta, r}) \right\|^2. \quad (44)$$

Lemma 4.10. *For every $j \in \{0, \dots, m-1\}$ and every $\kappa \in \mathbb{F}_q^n$,*

$$V_j(\kappa) = \sum_{\alpha^{\leq j} \in \mathcal{A}_{\leq j}} \prod_{r=0}^{2^j-1} |f(U^T k_{\alpha^{\leq j}, \kappa, r})|^2, \quad T_j(\kappa) = \sum_{\alpha^{\leq j} \in \mathcal{A}_{\leq j}} \left\| G(U^T k_{\alpha^{\leq j}, \kappa, 0}) \prod_{r=1}^{2^j-1} f(U^T k_{\alpha^{\leq j}, \kappa, r}) \right\|^2.$$

Proof. We will prove that these terms satisfy Eq. (42) by induction. The claim for $j = 0$ follows from Eq. (41).

Suppose that the statement holds at level $j-1$. We can uniquely decompose an element $\alpha^{\leq j} \in \mathcal{A}_{\leq j}$ into $(\alpha, \beta^L, \beta^R) \in H_j \times \mathcal{A}_{\leq j-1} \times \mathcal{A}_{\leq j-1}$ where we let

$$\alpha := \alpha_{j,0}, \quad \beta^L = (\beta_{h,a}^L := \alpha_{h,a})_{h,a}, \quad \beta^R = (\beta_{h,a}^R := \alpha_{h,a+2^{j-h-1}})_{h,a} \quad (45)$$

where the indices are over $1 \leq h < j$ and $0 \leq a < 2^{j-h-1}$.

By Eq. (43), for $0 \leq r < 2^{j-1}$, it is not hard to see that

$$k_{\alpha^{\leq j}, \kappa, r} = k_{\beta^L, \kappa + \alpha, r}, \quad k_{\alpha^{\leq j}, \kappa, 2^{j-1}+r} = k_{\beta^R, -\kappa - \alpha, r} \quad (46)$$

because of $(2^{j-1} + r) \bmod 2^h = r \bmod 2^h$, $\left\lfloor \frac{2^{j-1}+r}{2^h} \right\rfloor = 2^{j-h-1} + \left\lfloor \frac{r}{2^h} \right\rfloor$ for $h < j$ and $\varepsilon_{2^{j-1}+r} = -\varepsilon_r$.

Using Eq. (46) and the induction hypothesis, we obtain

$$\begin{aligned} & \sum_{\alpha^{\leq j} \in \mathcal{A}_{\leq j}} \prod_{r=0}^{2^j-1} |f(U^T k_{\alpha^{\leq j}, \kappa, r})|^2 \\ &= \sum_{\alpha \in H_j} \left(\sum_{\beta^L \in \mathcal{A}_{\leq j-1}} \prod_{r=0}^{2^{j-1}-1} |f(U^T k_{\beta^L, \kappa + \alpha, r})|^2 \right) \left(\sum_{\beta^R \in \mathcal{A}_{\leq j-1}} \prod_{r=0}^{2^{j-1}-1} |f(U^T k_{\beta^R, -\kappa - \alpha, r})|^2 \right) \\ &= \sum_{\alpha \in H_j} V_{j-1}(\kappa + \alpha) V_{j-1}(-\kappa - \alpha) = \sum_{\alpha \in H_j} V_{j-1}(\kappa + \alpha)^2 = V_j(\kappa). \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \left\| G(U^T k_{\alpha \leq j, \kappa, 0}) \prod_{r=1}^{2^j-1} f(U^T k_{\alpha \leq j, \kappa, r}) \right\|^2 &= \sum_{\alpha \in H_j} T_{j-1}(\kappa + \alpha) V_{j-1}(-\kappa - \alpha) \\ &= \sum_{\alpha \in H_j} T_{j-1}(\kappa + \alpha) V_{j-1}(\kappa + \alpha) = T_j(\kappa). \end{aligned}$$

Here we use the fact that V_{j-1} is even. This proves the two identities at level j . $\square$

We need the following upper bound of the sum of V 's and T 's. The proof of this lemma is rather involved. The tree structure is used in [Eq. \(49\)](#), which gives an upper bound of the first term as a product of discrete Gaussian masses. Then we use the known Gaussian masses to obtain the desired bound. For the second term, we use a trick like $xe^{-x} \leq ne^{-(1-1/n)x}$ to absorb the G terms in the exponent at the cost of small factor in the exponent. We then use the second bound of [Lemma 4.9](#) instead. The same proof strategy is used when proving [Lemma 4.16](#).

Lemma 4.11. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $2^{-\Omega(n)}$ over U , simultaneously for every $j \in \{1, \dots, m-1\}$,*

$$\sum_{\kappa \in R_j} V_j(\kappa) \leq 1 + 2^{((m-j-1)/2 - g/2 + o(1))n}, \quad \sum_{\kappa \in R_j} T_j(\kappa) \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{((m-j-1)/2 - g/2 + o(1))n}. \quad (47)$$

Proof. Recall $R_{j-1} = H_j \oplus R_j$ by definition ([Eq. \(38\)](#)) We define for $j \in \{1, \dots, m-1\}$

$$\bar{V}_j := \sum_{\kappa \in R_j} V_j(\kappa) = \sum_{\kappa \in R_{j-1}} V_{j-1}(\kappa)^2, \quad \bar{T}_j := \sum_{\kappa \in R_j} T_j(\kappa) = \sum_{\kappa \in R_{j-1}} T_{j-1}(\kappa) V_{j-1}(\kappa). \quad (48)$$

where the last equalities are obtained by summing [Eq. \(42\)](#) over $\kappa \in R_j$.

We first give a bound of $\bar{V}_j$, whose terms are products of 2^j factors of the form $|f(z)|^2$ for some z thanks to [Lemma 4.10](#). More precisely, for each summand indexed by κ and $\alpha \leq j$, we define

$$z_r := \varepsilon_r U^T k_{\alpha \leq j, \kappa, r}.$$

Using the fact that f is even and the above equation, we have

$$\left| f(U^T k_{\alpha \leq j, \kappa, r}) \right|^2 = |f(z_r)|^2 = \frac{1}{\rho_{1/s}(\mathcal{L})^2} \sum_{u_{r,0}, u_{r,1} \in Bz_r + q\mathcal{L}} e^{-\pi s^2 (\|u_{r,0}\|^2 + \|u_{r,1}\|^2) / q^2}$$

using the representation of f in [Eq. \(12\)](#). Plugging this to [Lemma 4.10](#) and expanding introduces vectors $u_{r,0}, u_{r,1} \in Bz_r + q\mathcal{L}$ for $r \in [0, 2^j)$ in the exponents, and the denominator $\rho_{1/s}(\mathcal{L})^{2^{j+1}}$. That is, the summand in $\bar{V}_j$ can be indexed by $(\alpha \leq j, \kappa)$ (from [Lemma 4.10](#)) and $(u_{r,b})_{r,b}$ (from the expansion).

Claim 4.12. *In the expansion of $\bar{V}_j$, the tuple $(u_{r,b})_{r,b}$ uniquely determines the summation indices $(\alpha \leq j, \kappa)$. Consequently, the expanded sum can be indexed by $(u_{r,b})_{r,b}$ without multiplicity.*

Proof. Since

$$B^{-1}u_{r,0} \bmod q = z_r = \varepsilon_r U^T k_{\alpha \leq j, \kappa, r},$$

the tuple $(u_{r,b})_{r,b}$ determines every leaf frequency via

$$k_{\alpha \leq j, \kappa, r} = \varepsilon_r U^{-T} (B^{-1}u_{r,0} \bmod q).$$

Given this, κ is determined by $k_{\alpha \leq j, \kappa, 0} = \kappa + \sum_{h=1}^j \alpha_{h,0}$ from Eq. (43); and $\varepsilon_0 = 1$; since $\kappa \in R_j$ and $\alpha_{h,0} \in H_h$, the R_j -component of this frequency is κ under the decomposition

$$\mathbb{F}_q^n = R_j \oplus H_1 \oplus \cdots \oplus H_j.$$

Similarly, $\alpha_{h,a}$ is determined by the H_h -component of $k_{\alpha \leq j, \kappa, r}$ for $r = 2^h a$, which is

$$\varepsilon_{r \bmod 2^h} \alpha_{h, \lfloor r/2^h \rfloor} = \varepsilon_{(2^h a) \bmod 2^h} \alpha_{h, \lfloor 2^h a/2^h \rfloor} = \varepsilon_0 \alpha_{h,a} = \alpha_{h,a}.$$

Thus all the variables κ and $\alpha_{h,a}$ are uniquely determined by $(u_{r,b})_{r,b}$. $\square$

We will simplify the exponent of each term. More precisely, we prove that

$$\sum_{r=0}^{2^j-1} \left(\|u_{r,0}\|^2 + \|u_{r,1}\|^2 \right) = 2^{j+1} \|a_{j,0}\|^2 + \frac{1}{2} \sum_{r=0}^{2^j-1} \|d_{0,r}\|^2 + \sum_{h=1}^j 2^{h-1} \sum_{r=0}^{2^{j-h}-1} \|d_{h,r}\|^2 \quad (49)$$

where $a_{j,0}$ and $(d_{h,r})_{h,r}$ are defined below. These vectors are uniquely determined by the vectors $u_{r,b}$, and the transformation is invertible. More precisely, for $r \in [0, 2^j)$, let

$$a_{0,r} := \frac{u_{r,0} + u_{r,1}}{2}, \quad d_{0,r} := u_{r,0} - u_{r,1}, \quad (50)$$

and, for $h \in \{1, \dots, j\}$ and $r \in [0, 2^{j-h})$, let

$$a_{h,r} := \frac{a_{h-1,2r} + a_{h-1,2r+1}}{2}, \quad d_{h,r} := a_{h-1,2r} - a_{h-1,2r+1}.$$

It is not hard to see that these transformations are invertible. Eq. (49) is proven by repeatedly applying

$$\|x_L\|^2 + \|x_R\|^2 = 2 \left\| \frac{x_L + x_R}{2} \right\|^2 + \frac{1}{2} \|x_L - x_R\|^2.$$

We call the map

$$(u_{r,b})_{\substack{r \in [0, 2^j) \\ b \in \{0,1\}}} \mapsto \left(a_{j,0}, (d_{h,r})_{\substack{h \in [0, j+1) \\ r \in [0, 2^{j-h})}} \right)$$

the *average-difference transformation*. The following claim provides the properties of this transformation that we need.

Claim 4.13. *The average-difference transformation extends to a linear bijection*

$$\Psi_j : (\mathbb{R}^n)^{[0, 2^j) \times \{0,1\}} \longrightarrow \mathbb{R}^n \times \prod_{h=0}^j (\mathbb{R}^n)^{[0, 2^{j-h})}.$$

When restricted to the tuples $(u_{r,b})_{r,b}$ occurring in the expansion of $\bar{V}_j$, this transformation has the following additional property.

- If all difference but $a_{j,0}$ are fixed, the possible values of $a_{j,0}$ are contained in a single coset of $\mathcal{L}$.
- Every $d_{0,r}$ belongs to K_0 .
- For every $h \in [j]$ and $r \in [0, 2^{j-h})$, if all lower-level differences $d_{\ell, r'}$ with $\ell < h$ are fixed, the possible values of $d_{h,r}$ are contained in a single coset of K_{h-1} .

Each coset may depend on the fixed variables.

Proof. At each node, the map

$$(x_L, x_R) \mapsto \left(\frac{x_L + x_R}{2}, x_L - x_R \right)$$

has inverse

$$(a, d) \mapsto \left(a + \frac{d}{2}, a - \frac{d}{2} \right),$$

which proves the bijection.

We now restrict to the tuples occurring in the expansion of $\overline{V}_j$. Fix all but one of the transformed variables and compare two possible values of the unfixed variable. If $a_{j,0}$ changes by x , the inverse transformation shows that every $u_{r,b}$ changes by x . Since every $u_{r,b}$ belongs to $\mathcal{L}$, we have $x \in \mathcal{L}$. Moreover, the two variables $u_{r,0}$ and $u_{r,1}$ come from the same coset of $q\mathcal{L}$, and hence

$$d_{0,r} = u_{r,0} - u_{r,1} \in q\mathcal{L} = K_0.$$

It remains to consider $d_{h,r}$ for $h \geq 1$. Recall the relation, for every t and b ,

$$B^{-1}u_{t,b} \bmod q = \varepsilon_t U^T k_{\alpha \leq j, \kappa, t}.$$

On the other hand, the definition of $d_{h,r} = a_{h-1,2r} - a_{h-1,2r+1}$ with the average transformations gives

$$2^h d_{h,r} = \sum_{i=0}^{2^{h-1}-1} \sum_{b \in \{0,1\}} \left(u_{t_i^L, b} - u_{t_i^R, b} \right),$$

where $t_i^L := 2^h r + i$, and $t_i^R := 2^h r + 2^{h-1} + i$. Combining these identities yields

$$B^{-1}(2^h d_{h,r}) \bmod q = U^T \sum_{i=0}^{2^{h-1}-1} \sum_{b \in \{0,1\}} \left(\varepsilon_{t_i^L} k_{\alpha \leq j, \kappa, t_i^L} - \varepsilon_{t_i^R} k_{\alpha \leq j, \kappa, t_i^R} \right).$$

Expanding the two k 's using [Eq. \(43\)](#), the κ -terms and the $\alpha_{\ell,a}$ -terms with $\ell \geq h$ have the same indices and coefficients and hence cancel. Every remaining term can be written as

$$\varepsilon_{a_L} \alpha_{\ell, a_L} - \varepsilon_{a_R} \alpha_{\ell, a_R}$$

for some $\ell < h$ and some a_L, a_R , where $\alpha_{\ell, a_L}, \alpha_{\ell, a_R} \in H_\ell$, so that this term belongs to H_ℓ . Thus the expression in parentheses belongs to

$$H_1 \oplus \cdots \oplus H_{h-1} = P_{h-1}.$$

Consequently,

$$B^{-1}(2^h d_{h,r}) \bmod q \in U^T P_{h-1} = S_{h-1}.$$

Since $2^h d_{h,r}$ is a sum of vectors in $\mathcal{L}$, this shows that $2^h d_{h,r} \in K_{h-1}$.

Fix all differences $d_{\ell,s}$ with $\ell < h$, and let v and v' be any two possible values of $d_{h,r}$ consistent with these fixed values. By the inverse transformation, the corresponding vectors $u_{t,b}$ change by a common lattice vector within each of the two groups defining $d_{h,r}$. Denoting these changes by $x_L, x_R \in \mathcal{L}$, we have

$$v' - v = x_L - x_R \in \mathcal{L}.$$

Moreover, the preceding containment gives $2^h v, 2^h v' \in K_{h-1}$, so $2^h(v' - v) \in K_{h-1}$. Since q is odd, multiplication by 2^h is invertible modulo q , and therefore $v' - v \in K_{h-1}$. Thus all possible values of $d_{h,r}$ lie in a single coset of K_{h-1} . $\square$

By the bijection and the uniqueness observation above, every term in the expansion of $\bar{V}_j$ is counted exactly once in the transformed variables. We first sum over $a_{j,0}$ and then over the variables $d_{h,r}$ in decreasing order of h . Their numbers, containing cosets, and Gaussian widths are as follows:

variable	number	containing coset	width
$a_{j,0}$	1	a coset of $\mathcal{L}$	$\frac{q}{2^{(j+1)/2}s}$
$d_{0,r}$	2^j	K_0	$\frac{\sqrt{2}q}{s}$
$d_{1,r}$	2^{j-1}	a coset of K_0	$\frac{q}{s}$
$d_{h+1,r}$	2^{j-h-1}	a coset of K_h	$\frac{q}{2^{h/2}s} = \sigma_h \quad (1 \leq h < j).$

The widths follow from the coefficients in [Eq. \(49\)](#). At each step, the values that occur in the current sum form a subset of one coset in the corresponding row. Extending the sum to the entire coset and applying [Lemma 2.1](#) therefore gives

$$\bar{V}_j \leq \frac{\rho_{q/(2^{(j+1)/2}s)}(\mathcal{L})}{\rho_{1/s}(\mathcal{L})^{2^{j+1}}} \rho_{\sqrt{2}q/s}(K_0)^{2^j} \rho_{q/s}(K_0)^{2^{j-1}} \prod_{h=1}^{j-1} \rho_{\sigma_h}(K_h)^{2^{j-h-1}}. \quad (51)$$

The denominator comes from the 2^j factors $|f(z_r)|^2$, each of which contributes $\rho_{1/s}(\mathcal{L})^{-2}$.

We remove the denominator using $\rho_{1/s}(\mathcal{L})^{2^{j+1}} \geq 1$, which only increases it. By [Lemma 3.3](#),

$$\rho_{q/(2^{(j+1)/2}s)}(\mathcal{L} \setminus \{0\}) \leq 2^{o(n)} \left(\frac{\beta^2 n q^2}{2\pi e \lambda^2 2^{j+1} s^2} \right)^{n/2} = 2^{o(n)} \left(\frac{t_0 q^2 d^2}{2^{j+2} t \lambda^2} \right)^{n/2} = 2^{((m-j-1)/2 - g + o(1))n},$$

where we use $t_0/t = 2^{-2g}$, $d/\lambda = 1 + O(1/n)$, and $\log_2 q = \frac{m+1}{2} + o(1)$. Since $K_0 = q\mathcal{L}$, [Lemma 3.1](#) gives

$$\rho_{\sqrt{2}q/s}(K_0) = \rho_{\sqrt{2}/s}(\mathcal{L}) \leq 1 + 2^{-(g-o(1))n}, \quad \rho_{q/s}(K_0) = \rho_{1/s}(\mathcal{L}) \leq 1 + 2^{-(1/2+g-o(1))n}.$$

Moreover, [Lemma 4.9](#) gives $\rho_{\sigma_h}(K_h) \leq 1 + 2^{-(g/2-o(1))n}$ for $1 \leq h \leq j-1$. Thus every remaining numerator mass in [Eq. \(51\)](#) is at most $1 + 2^{-(g/2-o(1))n}$. Multiplying all these bounds, we have the desired bound

$$\bar{V}_j \leq 1 + 2^{((m-j-1)/2 - g/2 + o(1))n}. \quad (52)$$

For $\bar{T}_j$, one of the factors $|f(z)|^2$ is replaced by $\|G(z)\|^2$. By the representation of G in [Eq. \(12\)](#),

$$G(z) = -\frac{2\pi s^2}{q\rho_{1/s}(\mathcal{L})} \sum_{u \in Bz + q\mathcal{L}} u e^{-\pi s^2 \|u\|^2 / q^2}.$$

Applying the Cauchy-Schwarz inequality gives

$$\begin{aligned} \|G(z)\|^2 &\leq \left(\frac{2\pi s^2}{q} \right)^2 \frac{1}{\rho_{1/s}(\mathcal{L})^2} \left(\sum_{u_0 \in Bz + q\mathcal{L}} \|u_0\|^2 e^{-\pi s^2 \|u_0\|^2 / q^2} \right) \left(\sum_{u_1 \in Bz + q\mathcal{L}} e^{-\pi s^2 \|u_1\|^2 / q^2} \right) \\ &= \left(\frac{2\pi s^2}{q} \right)^2 \frac{1}{\rho_{1/s}(\mathcal{L})^2} \sum_{u_0, u_1 \in Bz + q\mathcal{L}} \|u_0\|^2 e^{-\pi s^2 (\|u_0\|^2 + \|u_1\|^2) / q^2}. \end{aligned} \quad (53)$$

Thus the expansion for $\bar{T}_j$ is bounded by the expansion for $\bar{V}_j$ with an additional factor $(2\pi s^2/q)^2 \|u_{0,0}\|^2$ in each summand.

From the definitions of $a_{h,r}$ and $d_{h,r}$, we have $u_{0,0} = a_{j,0} + \frac{1}{2} \sum_{h=0}^j d_{h,0}$ which gives

$$\begin{aligned} \|u_{0,0}\|^2 &\leq (j+2) \left(\|a_{j,0}\|^2 + \frac{1}{4} \sum_{h=0}^j \|d_{h,0}\|^2 \right) \\ &\leq (j+1) \left(2^{j+1} \|a_{j,0}\|^2 + \frac{1}{2} \sum_{r=0}^{2^j-1} \|d_{0,r}\|^2 + \sum_{h=1}^j 2^{h-1} \sum_{r=0}^{2^{j-h}-1} \|d_{h,r}\|^2 \right) \end{aligned}$$

Note that all the terms in the right hand side appear in [Eq. \(49\)](#). We will use the inequality for all $R \geq 0$

$$Re^{-\pi s^2 R/q^2} \leq \frac{nq^2}{\pi s^2} e^{-\pi(1-1/n)s^2 R/q^2}. \quad (54)$$

because of $e^{x+y} \geq (1+x)e^y \geq xe^y$ with $x = \pi R s^2 / nq^2$. From this, we have

$$\begin{aligned} \|u_{0,0}\|^2 \exp \left(-\frac{\pi s^2 \left(2^{j+1} \|a_{j,0}\|^2 + \frac{1}{2} \sum_{r=0}^{2^j-1} \|d_{0,r}\|^2 + \sum_{h=1}^j 2^{h-1} \sum_{r=0}^{2^{j-h}-1} \|d_{h,r}\|^2 \right)}{q^2} \right) \\ \leq 2^{o(n)} \frac{nq^2}{\pi s^2} \exp \left(-\frac{\pi s^2 (1-1/n) \left(2^{j+1} \|a_{j,0}\|^2 + \frac{1}{2} \sum_{r=0}^{2^j-1} \|d_{0,r}\|^2 + \sum_{h=1}^j 2^{h-1} \sum_{r=0}^{2^{j-h}-1} \|d_{h,r}\|^2 \right)}{q^2} \right) \end{aligned}$$

where we use $j = 2^{o(n)}$.

The moment absorption multiplies every Gaussian width in the table above by $1/\sqrt{1-1/n}$, but does not change the summation domains or their containing cosets. Hence the same ordered summation argument based on [Claims 4.12](#) and [4.13](#) applies with these widened widths.

Note that the summand in which $a_{j,0} = 0$ and every $d_{h,r} = 0$ vanishes, since then $u_{0,0} = 0$. Hence, it suffices to bound the contribution in which at least one of these variables is nonzero.

We classify every transformed tuple other than the all-zero tuple as follows. Either all differences $d_{h,r}$ vanish and $a_{j,0} \neq 0$, or there is a smallest level h containing a nonzero difference; in the latter case, choose r such that $d_{h,r} \neq 0$. Thus $d_{\ell,r'} = 0$ for every $\ell < h$ and every r' .

In the first case, the tuple $u_{t,b} = 0$ for every t, b also occurs in the expansion and has all differences equal to zero. Hence the coset containing the possible values of $a_{j,0}$ contains zero and is therefore $\mathcal{L}$. Since $a_{j,0} \neq 0$, the corresponding sum is bounded by the Gaussian mass of $\mathcal{L} \setminus \{0\}$.

In the second case, if $h = 0$, then $d_{0,r} \in K_0 \setminus \{0\}$. Suppose that $h \geq 1$. The tuple $u_{t,b} = 0$ for every t, b has the same fixed lower-level differences and has $d_{h,r} = 0$. Therefore, the coset of K_{h-1} containing the possible values of $d_{h,r}$ contains zero and hence equals K_{h-1} . Since $d_{h,r} \neq 0$, we obtain $d_{h,r} \in K_{h-1} \setminus \{0\}$.

We bound all other variables by their full Gaussian masses as above and sum over at most $2^{j+1} = 2^{o(n)}$ choices of the distinguished nonzero variable. The second estimate in [Lemma 4.9](#) and [Lemma 3.3](#) then bound the Gaussian contribution by

$$2^{((m-j-1)/2 - g/2 + o(1))n}.$$

The prefactor from the primal representation of G , Cauchy-Schwarz, and moment absorption is at most $(s^2 \lambda / q)^2 2^{o(n)}$. Multiplying this by the preceding Gaussian-mass bound and absorbing all subexponential factors proves the second bound in [Eq. \(47\)](#). $\square$

The nonprincipal mass. As in [Lemma 4.5](#), let $v \in \mathcal{L}$ be a shortest vector, $w = B^{-1}v \bmod q$, $k_* = U^{-T}w$, and $\theta_* = (k_*)_{H_m}$. For $j \in \{0, \dots, m-1\}$, let

$$C_j^* := U^T k_* + S_j = w + S_j, \quad C_{j-1}^* \subseteq C_j^* \quad \text{for every } j \geq 1.$$

For $j \in \{0, \dots, m-1\}$, we define the following terms

$$V_j^* := |f(w)|^{2^{j+1}}, \quad B_j := V_j(k_*) - V_j^*, \quad T_j^* := \|G(w)\|^2 |f(w)|^{2(2^j-1)}, \quad E_j := T_j(k_*) - T_j^*, \quad (55)$$

where V_j^* and T_j^* are the principal terms in $V_j(k_*)$ and $T_j(k_*)$, respectively. Thus B_j and E_j are the sums of the remaining nonnegative terms. In particular, $U^T k_* = w$ and hence $B_0 = E_0 = 0$.

Since $k_* - \theta_* \in P_{m-1}$ and T_{m-1} is P_{m-1} -periodic, we have $T_{m-1}(k_*) = T_{m-1}(\theta_*)$. Plugging $j = m-1$ in [Eqs. \(44\)](#) and [\(55\)](#) shows that $T_{m-1}(k_*)$ is the sum over all α , and the term indexed by α^* is T_{m-1}^* . This gives the identity

$$E_{m-1} = \sum_{\alpha \neq \alpha^*} \left\| G(U^T k_{\alpha, \theta_*, 0}) \prod_{r=1}^{p-1} f(U^T k_{\alpha, \theta_*, r}) \right\|^2. \quad (56)$$

Our goal is to bound E_{m-1} . We prove the bound using induction using the recursive formula for B and E with some new terms. For $j \geq 1$, we define

$$\Delta_j := \sum_{\alpha \in H_j \setminus \{0\}} V_{j-1}(k_* + \alpha)^2, \quad \Gamma_j := \sum_{\alpha \in H_j \setminus \{0\}} T_{j-1}(k_* + \alpha) V_{j-1}(k_* + \alpha).$$

Separating the term $\alpha = 0$ in [Eq. \(42\)](#) gives

$$V_j(k_*) = V_{j-1}(k_*)^2 + \Delta_j, \quad T_j(k_*) = T_{j-1}(k_*) V_{j-1}(k_*) + \Gamma_j. \quad (57)$$

Since $V_j^* = (V_{j-1}^*)^2$, $T_j^* = T_{j-1}^* V_{j-1}^*$ by definition, subtracting these terms from [Eq. \(57\)](#) gives

$$B_j = 2V_{j-1}^* B_{j-1} + B_{j-1}^2 + \Delta_j, \quad E_j = E_{j-1}(V_{j-1}^* + B_{j-1}) + T_{j-1}^* B_{j-1} + \Gamma_j. \quad (58)$$

Lemma 4.14. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $o(1)$ over U , simultaneously for every $j \in \{1, \dots, m-1\}$,*

$$\Delta_j \leq 2^{-(1/2+g/4-o(1))n}, \quad \Gamma_j \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-(1/2+g/4-o(1))n}. \quad (59)$$

Proof. Fix $j \in \{1, \dots, m-1\}$. Under the bijection $\kappa + P_{j-1} \mapsto U^T \kappa + S_{j-1}$, the terms in Δ_j and Γ_j correspond to the S_{j-1} -cosets contained in C_j^* other than C_{j-1}^* . Conditional on $S_0, \dots, S_{j-1}$, each fixed such coset is contained in C_j^* with probability at most $\frac{q^{jn/m} - q^{(j-1)n/m}}{q^n - q^{(j-1)n/m}} \leq 2q^{-(n-jn/m)}$.

On the event $w \notin S_{m-1}$, the zero coset is not contained in C_j^* . Since $V_{j-1}(0) \geq 1$, linearity of expectation and the definitions in [Eq. \(48\)](#) give

$$\begin{aligned} \mathbb{E} [\mathbf{1}_{\{w \notin S_{m-1}\}} \Delta_j \mid S_0, \dots, S_{j-1}] &\leq 2q^{-(n-jn/m)} (\bar{V}_j - 1) \leq 2^{-(1/2+g/2+(m-j)/(2m)-o(1))n}, \\ \mathbb{E} [\mathbf{1}_{\{w \notin S_{m-1}\}} \Gamma_j \mid S_0, \dots, S_{j-1}] &\leq 2q^{-(n-jn/m)} \bar{T}_j \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-(1/2+g/2+(m-j)/(2m)-o(1))n}, \end{aligned}$$

where we use [Lemma 4.11](#) and $\log_2 q = (m+1)/2 + o(1)$ in the final inequalities. Markov's inequality shows that either bound in [Eq. \(59\)](#) fails for a fixed j with probability at most

$$2^{-(g/4+(m-j)/(2m)-o(1))n}.$$

Finally, $\Pr[w \in S_{m-1}] = \frac{q^{n-n/m}-1}{q^n-1} = o(1)$. A union bound shows that all the above events occur except with probability $o(1)$. $\square$

Proof of Lemma 4.5. We will bound the term E_{m-1} in [Eq. \(56\)](#). By [Lemma 4.14](#), the conclusion of that lemma holds simultaneously for every $j \in \{1, \dots, m-1\}$ except with probability $o(1)$.

By [Lemma 2.1](#), $|f(w)| = \frac{\rho_{1/s}(\mathcal{L}+v/q)}{\rho_{1/s}(\mathcal{L})} \leq 1$, and hence $V_j^* = |f(w)|^{2^{j+1}} \leq 1$. [Lemma 3.4](#) and $q = 2^{o(n)}$ give

$$\|G(w)\| \leq |f(w)| \left(\frac{2\pi s^2 \lambda}{q} + 2^{-\Omega(n)} s^2 \lambda \right) \leq O\left(\frac{s^2 \lambda}{q} \right) |f(w)|.$$

This gives $T_j^* = \|G(w)\|^2 |f(w)|^{2(2^j-1)} \leq O\left(\frac{s^2\lambda}{q}\right)^2 |f(w)|^{2^{j+1}} \leq O\left(\frac{s^2\lambda}{q}\right)^2$.

We first bound B_j . Starting with $B_0 = 0$, the first recurrence in [Eq. \(58\)](#) gives inductively

$$B_j \leq 4^j 2^{-(1/2+g/4-o(1))n} = o(1), \quad (60)$$

which can be verified by $2B_{j-1} + B_{j-1}^2 + 2^{-(1/2+g/4-o(1))n} \leq 4^j 2^{-(1/2+g/4-o(1))n}$. Since $j \leq m = O(\log n)$, the factor 4^j is $2^{o(n)}$. Using this bound in the second recurrence in [Eq. \(58\)](#) gives, together with $V_{j-1}^* + B_{j-1} \leq 2$ for all sufficiently large n because of [Eq. \(60\)](#) and $V_{j-1}^* = |f(w)|^{2^j} \leq 1$,

$$E_j \leq \left(\frac{s^2\lambda}{q}\right)^2 2^{-(1/2+g/4-o(1))n}$$

for every $j \leq m-1$. Iterating the recurrence gives a factor $2^{O(j)} = 2^{o(n)}$, which is absorbed by the $o(1)$ term.

Plugging $j = m-1$ proves [Lemma 4.5](#) given $g > 0$. $\square$

The second moment. For $\mathbf{i} \in [N]^p$ and $\theta \in H_m$, let

$$\mathcal{K}_{\mathbf{i}}(\theta) := Q^{p-1} C_{\delta}(\mathbf{i}) 2\pi i X_{0,i_0} \omega_q^{(\theta, (Y_{m-1,0}(\mathbf{i}))_{H_m})}.$$

Then we have

$$\widehat{A}_{\delta}(\theta) = \frac{1}{N^p} \sum_{\mathbf{i} \in [N]^p} \mathcal{K}_{\mathbf{i}}(\theta), \quad \mathbb{E}_X[\mathcal{K}_{\mathbf{i}}(\theta) \mid U, \delta] = A_{\delta}(\theta). \quad (61)$$

For $\mathbf{i}, \mathbf{i}' \in [N]^p$, let $I(\mathbf{i}, \mathbf{i}') := \{r \in [0, p) : i_r = i'_r\}$. Expanding the squared norm in [Eq. \(61\)](#) gives

$$\mathbb{E}_{\delta, X} \left[\left\| \widehat{A}_{\delta}(\theta) - A_{\delta}(\theta) \right\|^2 \mid U \right] = \frac{1}{N^{2p}} \sum_{\mathbf{i}, \mathbf{i}' \in [N]^p} \mathbb{E}_{\delta, X} \left[(\mathcal{K}_{\mathbf{i}}(\theta) - A_{\delta}(\theta))^* (\mathcal{K}_{\mathbf{i}'}(\theta) - A_{\delta}(\theta)) \mid U \right]. \quad (62)$$

We use the following bound for every term in this sum, whose proof is deferred to the end of this section.

Lemma 4.15. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $2^{-\Omega(n)}$ over U , the following hold for every $\theta \in H_m$ and every $\mathbf{i}, \mathbf{i}' \in [N]^p$: the expectation in [Eq. \(62\)](#) is zero if $I(\mathbf{i}, \mathbf{i}') = \emptyset$, and otherwise its absolute value is at most*

$$\left(\frac{s^2\lambda}{q}\right)^2 Q^{|I(\mathbf{i}, \mathbf{i}')|-1} 2^{o(n)}. \quad (63)$$

Proof of Lemma 4.6. We work on the event in [Lemma 4.15](#). For every nonempty $I \subseteq [0, p)$, there are at most $N^{2p-|I|}$ ordered pairs $(\mathbf{i}, \mathbf{i}')$ satisfying $I(\mathbf{i}, \mathbf{i}') = I$. Therefore, [Eqs. \(62\) and \(63\)](#) give

$$\begin{aligned} \mathbb{E}_{\delta, X} \left[\left\| \widehat{A}_{\delta}(\theta) - A_{\delta}(\theta) \right\|^2 \mid U \right] &\leq \frac{1}{N^{2p}} \sum_{\emptyset \neq I \subseteq [0, p)} N^{2p-|I|} \left(\frac{s^2\lambda}{q}\right)^2 Q^{|I|-1} 2^{o(n)} \\ &= \left(\frac{s^2\lambda}{q}\right)^2 Q^{-1} 2^{o(n)} \left[\left(1 + \frac{Q}{N}\right)^p - 1 \right] \leq \left(\frac{s^2\lambda}{q}\right)^2 Q^{-1} 2^{o(n)}, \end{aligned}$$

where the last inequality follows from $N = Qp^4$. The event in [Lemma 4.15](#) is uniform in θ , so the conclusion holds simultaneously for every $\theta \in H_m$. $\square$

The first lemma bounds the sums of V_j and T_j over the affine space $\kappa + H_{j+1}$. The proof uses the same tricks as in [Lemma 4.11](#).

Lemma 4.16. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $2^{-\Omega(n)}$ over U , simultaneously for every $j \in \{0, \dots, m-2\}$ and every $\kappa \in \mathbb{F}_q^n$,*

$$\sum_{\alpha \in H_{j+1}} V_j(\kappa + \alpha) \leq 1 + 2^{-\Omega(n)}, \quad \sum_{\alpha \in H_{j+1}} T_j(\kappa + \alpha) \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{o(n)}. \quad (64)$$

Proof. While the same strategy will apply, we first focus on $j = 0$ for the intuition. By [Lemma 4.10](#), the terms in the statement become

$$\sum_{\alpha \in H_1} V_0(\kappa + \alpha) = \sum_{\alpha \in H_1} |f(U^T(\kappa + \alpha))|^2 \quad \text{and} \quad \sum_{\alpha \in H_1} T_0(\kappa + \alpha) = \sum_{\alpha \in H_1} \|G(U^T(\kappa + \alpha))\|^2.$$

Expanding the first representations in [Eqs. \(9\) and \(10\)](#) and using [Eq. \(49\)](#) recursively gives

$$\sum_{\alpha \in H_1} V_0(\kappa + \alpha) \leq \frac{\rho_{\sigma_1}(K_1 + z)}{\rho_{1/s}(\mathcal{L})^2} \rho_{\sqrt{2}q/s}(K_0) \leq \frac{\rho_{\sigma_1}(K_1)}{\rho_{1/s}(\mathcal{L})^2} \rho_{\sqrt{2}q/s}(K_0) \quad (65)$$

for some shift $z \in \mathbb{R}^n$, and the last inequality is due to [Lemma 2.1](#).

We do the same calculation for $j \geq 1$. By [Lemma 4.10](#), the two sums in the statement are

$$\sum_{\alpha \in H_{j+1}} V_j(\kappa + \alpha) = \sum_{\alpha \in H_{j+1}} \sum_{\beta \leq j \in \mathcal{A}_{\leq j}} \prod_{r=0}^{2^j-1} |f(U^T k_{\beta \leq j, \kappa + \alpha, r})|^2$$

and

$$\sum_{\alpha \in H_{j+1}} T_j(\kappa + \alpha) = \sum_{\alpha \in H_{j+1}} \sum_{\beta \leq j \in \mathcal{A}_{\leq j}} \left\| G(U^T k_{\beta \leq j, \kappa + \alpha, 0}) \prod_{r=1}^{2^j-1} f(U^T k_{\beta \leq j, \kappa + \alpha, r}) \right\|^2. \quad (66)$$

We first expand the scalar sum. For each $\alpha \in H_{j+1}$, $\beta \leq j \in \mathcal{A}_{\leq j}$, and $r \in [0, 2^j)$, set

$$z_r := \varepsilon_r U^T k_{\beta \leq j, \kappa + \alpha, r}.$$

By the evenness of f and its primal representation,

$$|f(U^T k_{\beta \leq j, \kappa + \alpha, r})|^2 = |f(z_r)|^2 = \frac{1}{\rho_{1/s}(\mathcal{L})^2} \sum_{u_{r,0}, u_{r,1} \in Bz_r + q\mathcal{L}} \exp\left(-\frac{\pi s^2}{q^2} (\|u_{r,0}\|^2 + \|u_{r,1}\|^2)\right).$$

Expanding the product therefore is decomposed by the summand indexed by $u_{r,0}, u_{r,1} \in Bz_r + q\mathcal{L}$ for $r \in [0, 2^j)$, that is, one with the Gaussian factor

$$\frac{1}{\rho_{1/s}(\mathcal{L})^{2^{j+1}}} \exp\left(-\frac{\pi s^2}{q^2} \sum_{r=0}^{2^j-1} (\|u_{r,0}\|^2 + \|u_{r,1}\|^2)\right).$$

For the corresponding average-difference variables, [Eq. \(49\)](#) gives

$$\sum_{r=0}^{2^j-1} (\|u_{r,0}\|^2 + \|u_{r,1}\|^2) = 2^{j+1} \|a_{j,0}\|^2 + \frac{1}{2} \sum_{r=0}^{2^j-1} \|d_{0,r}\|^2 + \sum_{h=1}^j 2^{h-1} \sum_{r=0}^{2^{j-h}-1} \|d_{h,r}\|^2.$$

The Gaussian widths in the table below follow from these coefficients.

The proofs of [Claims 4.12 and 4.13](#) give the following variants for the present fixed- κ sum.

Claim 4.17. Fix $\kappa \in \mathbb{F}_q^n$. In the expansion of $\sum_{\alpha \in H_{j+1}} V_j(\kappa + \alpha)$, the tuple $(u_{r,b})_{r,b}$ uniquely determines $(\alpha, \beta^{\leq j})$. Consequently, the expanded sum can be indexed by $(u_{r,b})_{r,b}$ without multiplicity.

Sketch of proof. As in the proof of [Claim 4.12](#), the tuple $(u_{r,b})_{r,b}$ determines every $k_{\beta^{\leq j}, \kappa + \alpha, r}$. Since κ is fixed, the direct-sum decomposition gives $\alpha = (k_{\beta^{\leq j}, \kappa + \alpha, 0} - \kappa)_{H_{j+1}}$, and, for $h \in [j]$ and $a \in [0, 2^{j-h})$, we also have $\beta_{h,a} = (k_{\beta^{\leq j}, \kappa + \alpha, 2^h a} - \varepsilon_{2^h a} \kappa)_{H_h}$. Thus $(u_{r,b})_{r,b}$ uniquely determines $(\alpha, \beta^{\leq j})$. $\square$

Claim 4.18. Fix $\kappa \in \mathbb{F}_q^n$. The average-difference transformation is a bijection between the tuples $(u_{r,b})_{r,b}$ occurring in the expansion of $\sum_{\alpha \in H_{j+1}} V_j(\kappa + \alpha)$ and the transformed tuples arising from them. These transformed tuples satisfy the following properties:

- After all difference variables are fixed, the possible values of $a_{j,0}$ are contained in a single coset of K_{j+1} .
- Every $d_{0,r}$ belongs to K_0 .
- For every $h \in [j]$ and $r \in [0, 2^{j-h})$, after all lower-level differences $d_{\ell,r'}$ with $\ell < h$ are fixed, the possible values of $d_{h,r}$ are contained in a single coset of K_{h-1} .

Each coset may depend on the fixed variables.

Sketch of proof. Bijectivity follows from the bijectivity of Ψ_j . Suppose that all difference variables are fixed and that the final average changes by x . Then every $u_{r,b}$ changes by the same vector $x \in \mathcal{L}$. Comparing the residues at $r = 0$ gives

$$B^{-1}x \bmod q \in U^T(H_1 \oplus \cdots \oplus H_{j+1}) = S_{j+1},$$

and hence $x \in K_{j+1}$. The assertions for the difference variables follow exactly as in [Claim 4.13](#); the additional $\alpha \in H_{j+1}$ cancels from the two child blocks. $\square$

By the two claims above, the expanded sum can be indexed by the transformed variables without multiplicity. Their numbers, containing cosets, and Gaussian widths are summarized below. For $a_{j,0}$, all difference variables are fixed. For $d_{h,r}$ with $h \geq 1$, all lower-level differences are fixed.

variable	number	containing coset	width
$a_{j,0}$	1	a coset of K_{j+1}	$\frac{q}{2^{(j+1)/2}s} = \sigma_{j+1}$
$d_{0,r}$	2^j	K_0	$\frac{\sqrt{2}q}{s}$
$d_{1,r}$	2^{j-1}	a coset of K_0	$\frac{q}{s}$
$d_{h+1,r}$	2^{j-h-1}	a coset of K_h	$\frac{q}{2^{h/2}s} = \sigma_h \quad (1 \leq h < j).$

We first sum over $a_{j,0}$ while all difference variables are fixed. The first item of [Claim 4.18](#) and [Lemma 2.1](#) bound this sum by $\rho_{\sigma_{j+1}}(K_{j+1})$. We then sum over the variables $d_{h,r}$ in decreasing order of h . When $d_{h,r}$ is summed, all lower-level differences remain fixed, so the third item of [Claim 4.18](#) applies. Finally, every $d_{0,r}$ belongs to K_0 . Applying [Lemma 2.1](#) at each step gives

$$\sum_{\alpha \in H_{j+1}} V_j(\kappa + \alpha) \leq \frac{\rho_{\sigma_{j+1}}(K_{j+1})}{\rho_{1/s}(\mathcal{L})^{2^{j+1}}} \rho_{\sqrt{2}q/s}(K_0)^{2^j} \rho_{q/s}(K_0)^{2^{j-1}} \prod_{h=1}^{j-1} \rho_{\sigma_h}(K_h)^{2^{j-h-1}}. \quad (67)$$

By [Lemma 4.9](#),

$$\rho_{\sigma_h}(K_h) = 1 + \rho_{\sigma_h}(K_h \setminus \{0\}) \leq 1 + 2^{-(g/2 - o(1))n}$$

for every $h \in \{1, \dots, j+1\}$. Moreover, since $K_0 = q\mathcal{L}$, we have

$$\begin{aligned} \rho_{\sqrt{2}q/s}(K_0) &= \rho_{\sqrt{2}/s}(\mathcal{L}) \leq 1 + 2^{-(g - o(1))n}, \\ \rho_{q/s}(K_0) &= \rho_{1/s}(\mathcal{L}) \leq 1 + 2^{-(1/2 + g - o(1))n} \leq 1 + 2^{-(g - o(1))n}, \end{aligned}$$

by [Lemmas 3.1](#) and [3.3](#). Since the denominator is at least one and the total multiplicity of the numerator factors is at most $2^{j+1} \leq 2p = 2^{o(n)}$, the right-hand sides of [Eqs. \(65\)](#) and [\(67\)](#) are at most

$$\left(1 + 2^{-(g/2 - o(1))n}\right)^{2^{o(n)}} = 1 + 2^{-\Omega(n)}.$$

For [Eq. \(66\)](#), let $z_r := U^T k_{\beta \leq j, \kappa + \alpha, r}$ to simplify the expression. The first representation of G gives

$$\|G(z_0)\|^2 = \left(\frac{2\pi s^2}{q\rho_{1/s}(\mathcal{L})}\right)^2 \sum_{u_{0,0}, u_{0,1} \in Bz_0 + q\mathcal{L}} \langle u_{0,0}, u_{0,1} \rangle e^{-\pi s^2(\|u_{0,0}\|^2 + \|u_{0,1}\|^2)/q^2}.$$

Here $u_{0,0}$ and $u_{0,1}$ are the variables from the two copies of $G(z_0)$ in the norm.

The remainder of the argument is the same as in the proof of the second bound in [Lemma 4.11](#). We first bound the inner product by

$$|\langle u_{0,0}, u_{0,1} \rangle| \leq \frac{1}{2} (\|u_{0,0}\|^2 + \|u_{0,1}\|^2) \leq \frac{1}{2} \sum_{r=0}^{2^j-1} (\|u_{r,0}\|^2 + \|u_{r,1}\|^2).$$

Then, we apply the following inequality followed by [Eq. \(49\)](#) to the exponent:

$$\frac{R}{2} e^{-\pi s^2 R/q^2} \leq \frac{nq^2}{2\pi s^2} e^{-\pi(1-1/n)s^2 R/q^2} \leq \frac{nq^2}{\pi s^2} e^{-\pi(1-1/n)s^2 R/q^2}.$$

for $R = \sum_{r=0}^{2^j-1} (\|u_{r,0}\|^2 + \|u_{r,1}\|^2)$. For $j = 0$, the same calculation gives the right-hand side of [Eq. \(65\)](#) with σ_1 and $\sqrt{2}q/s$ replaced by $\sigma_1/\sqrt{1-1/n}$ and $\sqrt{2}q/(s\sqrt{1-1/n})$, respectively, and multiplied by $(2\pi s^2/q)^2 nq^2/(\pi s^2)$.

For $j \geq 1$, summing the resulting Gaussian terms and applying [Lemma 2.1](#) gives

$$\begin{aligned} \sum_{\alpha \in H_{j+1}} T_j(\kappa + \alpha) &\leq \left(\frac{2\pi s^2}{q}\right)^2 \frac{nq^2}{\pi s^2} \frac{\rho_{\sigma_{j+1}/\sqrt{1-1/n}}(K_{j+1})}{\rho_{1/s}(\mathcal{L})^{2^{j+1}}} \\ &\quad \cdot \rho_{\sqrt{2}q/(s\sqrt{1-1/n})}(K_0)^{2^j} \rho_{q/(s\sqrt{1-1/n})}(K_0)^{2^{j-1}} \prod_{h=1}^{j-1} \rho_{\sigma_h/\sqrt{1-1/n}}(K_h)^{2^{j-h-1}}. \end{aligned} \quad (68)$$

The second estimate in [Lemma 4.9](#) bounds the factors involving $\rho_{\sigma_h/\sqrt{1-1/n}}(K_h)$. Moreover, given $K_0 = q\mathcal{L}$, we have $\rho_{\sqrt{2}q/(s\sqrt{1-1/n})}(K_0) = \rho_{\sqrt{2}/(s\sqrt{1-1/n})}(\mathcal{L})$, which is bounded by [Lemma 3.3](#). Hence the product of Gaussian masses in [Eq. \(68\)](#), including its denominator, is at most $2^{o(n)}$. Consequently,

$$\sum_{\alpha \in H_{j+1}} T_j(\kappa + \alpha) \leq \left(\frac{2\pi s^2}{q}\right)^2 \frac{nq^2}{\pi s^2} 2^{o(n)} = \left(\frac{s^2 \lambda}{q}\right)^2 \frac{4\pi nq^2}{s^2 \lambda^2} 2^{o(n)} \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{o(n)},$$

where we use $s^2 \lambda^2 = \Theta(n)$ and $q = n^{O(1)}$ in the last inequality. This proves the second bound in [Eq. \(64\)](#). $\square$

The next lemma bounds the same sums after the factors indexed by a nonempty set I have been removed. We introduce some notations. For $j \in \{0, \dots, m-1\}$, $\kappa \in \mathbb{F}_q^n$, and $I \subseteq [0, 2^j]$, let

$$V_{j,I}(\kappa) := \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \prod_{r \in [0, 2^j] \setminus I} |f(U^T k_{\alpha \leq j, \kappa, r})|^2.$$

We similarly define the notation for T , where we require $0 \notin I$ because the term indexed by 0 is $G(\cdot)$ and is retained in T . For $0 \notin I$, we define

$$T_{j,I}(\kappa) := \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \left\| G(U^T k_{\alpha \leq j, \kappa, 0}) \prod_{r \in [1, 2^j] \setminus I} f(U^T k_{\alpha \leq j, \kappa, r}) \right\|^2.$$

In particular, $V_{j, \emptyset} = V_j$ and $T_{j, \emptyset} = T_j$.

Lemma 4.19. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. On the event in [Lemma 4.16](#), simultaneously for every $j \in \{0, \dots, m-1\}$, every $\kappa \in \mathbb{F}_q^n$, and every nonempty $I \subseteq [0, 2^j)$,*

$$V_{j,I}(\kappa) \leq Q^{|I|-1} 2^{o(n)}, \quad T_{j,I}(\kappa) \leq \left(\frac{s^2 \lambda}{q} \right)^2 Q^{|I|-1} 2^{o(n)} \quad \text{if } 0 \notin I. \quad (69)$$

Proof. We use induction on j . For $j = 0$, the only nonempty set is $I = \{0\}$, and $V_{0,\{0\}}(\kappa) = 1$. The second bound does not apply, and will not be used in the later induction.

Suppose that $j \geq 1$, and let $I_L := I \cap [0, 2^{j-1})$, $I_R := \{r \in [0, 2^{j-1}) : 2^{j-1} + r \in I\}$. By the decomposition used in [Lemma 4.10](#) and the inductive notions in [Eqs. \(45\) and \(46\)](#), we have

$$V_{j,I}(\kappa) = \sum_{\alpha \in H_j} V_{j-1,I_L}(\kappa + \alpha) V_{j-1,I_R}(-\kappa - \alpha), \quad T_{j,I}(\kappa) = \sum_{\alpha \in H_j} T_{j-1,I_L}(\kappa + \alpha) V_{j-1,I_R}(-\kappa - \alpha),$$

where the second identity is used only when $0 \notin I$.

We first prove the bound for $V_{j,I}(\kappa)$. If exactly one of I_L, I_R is nonempty, the induction hypothesis bounds the corresponding factor for each α , while the first bound in [Lemma 4.16](#) bounds the sum of the other factor over $\alpha \in H_j$. If both sets are nonempty, the induction hypothesis bounds both factors for each α , and summing over $|H_j| = Q$ choices of α gives

$$Q \cdot Q^{|I_L|-1} Q^{|I_R|-1} = Q^{|I|-1}.$$

This proves the first inequality of [Eq. \(69\)](#).

Now suppose that $0 \notin I$, and hence $0 \notin I_L$. If $I_L \neq \emptyset$ and $I_R = \emptyset$, the induction hypothesis bounds the first factor in the second recurrence, and the first bound in [Lemma 4.16](#) bounds the sum of the second factor. If $I_L = \emptyset$ and $I_R \neq \emptyset$, the second bound in [Lemma 4.16](#) bounds the sum of the first factor, and the first inequality of [Eq. \(69\)](#) bounds the second factor for each α . If both sets are nonempty, the induction hypotheses bound both factors for each α , and summing over $\alpha \in H_j$ gives

$$Q \cdot \left(\frac{s^2 \lambda}{q} \right)^2 Q^{|I_L|-1} Q^{|I_R|-1} = \left(\frac{s^2 \lambda}{q} \right)^2 Q^{|I|-1}.$$

This proves the second inequality of [Eq. \(69\)](#).

The suppressed factor $2^{o(n)}$ should be carefully proven. Observe that in the scalar induction we use the first bound in [Lemma 4.16](#) at most $2^j \leq p = 2^{o(n)}$ times. Their product is $(1 + 2^{-\Omega(n)})^p = 1 + 2^{-\Omega(n)}$ given $p = 2^{o(n)}$. In the induction for $T_{j,I}$, the second bound in [Lemma 4.16](#) is used at most once: when $I_L = \emptyset$, it directly bounds the sum of $T_{j-1}(\kappa + \alpha)$ over $\alpha \in H_j$, so no further induction involving T is needed. All other factors are the factors discussed in the bound of the first term. Thus the total suppressed factor is $2^{o(n)}$. This proves the lemma. $\square$

Now we are ready to prove, using the above lemmas, [Lemma 4.15](#) which states that

$$|\mathbb{E}_{\delta, X} [(\mathcal{K}_i(\theta) - A_\delta(\theta))^* (\mathcal{K}_{i'}(\theta) - A_\delta(\theta)) \mid U]| \leq \left(\frac{s^2 \lambda}{q} \right)^2 Q^{|I(i, i')|-1} 2^{o(n)}. \quad (70)$$

for $I(i, i') \neq \emptyset$ and it being zero otherwise, where

$$\mathcal{K}_i(\theta) := Q^{p-1} C_\delta(i) 2\pi i X_{0, i_0} \omega_q^{(\theta, (Y_{m-1, 0}(i))_{H_m})}.$$

Proof of Lemma 4.15. We assume that all of the bounds in Lemmas 4.11 and 4.16 hold, which happens except with probability $2^{-\Omega(n)}$ over U . Fix $\theta \in H_m$ and $\mathbf{i}, \mathbf{i}' \in [N]^p$, and let $I := I(\mathbf{i}, \mathbf{i}')$.

We first consider the case $I = \emptyset$. In this case, $i_r \neq i'_r$ for every $r \in [0, p)$. For fixed U and δ , the definitions of $C_\delta(\mathbf{i})$ and $Y_{m-1,0}(\mathbf{i})$ show that $\mathcal{K}_\mathbf{i}(\theta)$ is determined by $(X_{r,i_r})_{r \in [0,p)}$. Similarly, $\mathcal{K}_{\mathbf{i}'}(\theta)$ is determined by $(X_{r,i'_r})_{r \in [0,p)}$. These two collections of samples are independent. Both random variables have conditional expectation $A_\delta(\theta)$. Their independence therefore gives

$$\mathbb{E}_X [(\mathcal{K}_\mathbf{i}(\theta) - A_\delta(\theta))^* (\mathcal{K}_{\mathbf{i}'}(\theta) - A_\delta(\theta)) \mid U, \delta] = 0$$

Averaging over δ proves the claim for $I = \emptyset$.

Suppose that $I \neq \emptyset$. We apply Eq. (26) to the entire scalar term

$$Q^{p-1} C_\delta(\mathbf{i}) \omega_q^{\langle \theta, (Y_{m-1,0}(\mathbf{i}))_{H_m} \rangle}$$

and similarly to the term indexed by $\mathbf{i}'$. For each α , we index the terms in the resulting summand by

$$\omega_q^{\langle k_{\alpha, \theta, r}, Y_{r, i_r} \rangle}$$

by r for every $r \in [0, p)$. The term with $r = 0$ is additionally multiplied by $2\pi i X_{0, i_0}$.

Averaging the product over δ , character orthogonality eliminates all terms indexed by $\alpha \neq \alpha'$ and only the terms with $\alpha = \alpha'$ remain. After averaging over δ , fix a remaining α and $r \in [1, p)$. The terms involving the r -th samples in $\mathcal{K}_\mathbf{i}(\theta)^* \mathcal{K}_{\mathbf{i}'}(\theta)$ are

$$\overline{\omega_q^{\langle k_{\alpha, \theta, r}, Y_{r, i_r} \rangle}} \omega_q^{\langle k_{\alpha, \theta, r}, Y_{r, i'_r} \rangle}.$$

If $r \notin I$, then $i_r \neq i'_r$, so X_{r, i_r} and X_{r, i'_r} , and hence Y_{r, i_r} and Y_{r, i'_r} , are independent. Taking the expectation over these two samples gives

$$\overline{f(U^T k_{\alpha, \theta, r})} f(U^T k_{\alpha, \theta, r}) = |f(U^T k_{\alpha, \theta, r})|^2.$$

If $r \in I$, then $i_r = i'_r$, so the two character terms are evaluated at the same Y_{r, i_r} and their product is

$$\overline{\omega_q^{\langle k_{\alpha, \theta, r}, Y_{r, i_r} \rangle}} \omega_q^{\langle k_{\alpha, \theta, r}, Y_{r, i_r} \rangle} = 1.$$

If $r = 0$, the resulting term is $\|G(U^T k_{\alpha, \theta, 0})\|^2$ if $0 \notin I$ and $\mathbb{E}_{X \sim D_{\mathcal{L}^*, s}}[\|2\pi X\|^2]$ if $0 \in I$. Consequently,

$$\mathbb{E}_{\delta, X} [\mathcal{K}_\mathbf{i}(\theta)^* \mathcal{K}_{\mathbf{i}'}(\theta) \mid U] = \begin{cases} T_{m-1, I}(\theta), & 0 \notin I, \\ \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}}[\|2\pi X\|^2] V_{m-1, I}(\theta), & 0 \in I. \end{cases} \quad (71)$$

By Lemma 3.2, $s^2 \lambda^2 = \Theta(n)$, and $q = n^{O(1)}$, we have

$$\mathbb{E}_{X \sim D_{\mathcal{L}^*, s}}[\|2\pi X\|^2] \leq O(ns^2) \leq (s^2 \lambda / q)^2 2^{o(n)}.$$

Therefore, Lemma 4.19 bounds the absolute value of Eq. (71) by $(s^2 \lambda / q)^2 Q^{|I|-1} 2^{o(n)}$.

Applying the same character expansion to two independent copies gives

$$\mathbb{E}_\delta [\|A_\delta(\theta)\|^2 \mid U] = T_{m-1}(\theta) \leq \sum_{\kappa \in R_{m-1}} T_{m-1}(\kappa) \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-(g/2 - o(1))n}, \quad (72)$$

where we used $R_{m-1} = H_m$ and Lemma 4.11. Since the conditional expectation of every $\mathcal{K}_\mathbf{i}(\theta)$ is $A_\delta(\theta)$, the expectation in the statement is the difference between Eq. (71) and Eq. (72). Since $I \neq \emptyset$, the triangle inequality proves the desired bound, after absorbing a factor of two into $2^{o(n)}$. $\square$

5 CVP with a distance guarantee

This section proves the following result.

Theorem 5.1. *For every fixed constant $0 < \alpha < \frac{1}{2\sqrt{t_0}} = 1.03925\dots$, there is a randomized classical algorithm that solves CVP with probability at least $2/3$ given a lattice $\mathcal{L}$ and a target $y \in \mathbb{R}^n$ promised that $\text{dist}(y, \mathcal{L}) \leq \alpha \lambda_1(\mathcal{L})$ in worst-case time and space $2^{n/2+o(n)}$.*

We use y as the target vector and $\alpha < \frac{1}{2\sqrt{t_0}}$ in this section throughout. We will use the same parameters as the previous section. Formally, p is the largest power of two satisfying $p \leq n/\log^3 n$ and q is the smallest prime larger than $2\sqrt{p}$. Let $m = \log_2 p + 1$, $Q = q^{n/m} = 2^{n/2+o(n)}$ and $N = Qp^4 = 2^{n/2+o(n)}$.

5.1 The first estimation target

Let $v \in \mathcal{L}$ be closest to y and let $e = v - y$. We define the shifted versions of f and G as follows for $w \in \mathbb{F}_q^n$:

$$\begin{aligned} f_y(w) &:= F_{1/s} \left(\frac{Bw - y}{q} \right) = \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[\omega_q^{\langle w, [X]_q \rangle} e^{-2\pi i \langle X, y \rangle / q} \right], \\ G_y(w) &:= \nabla F_{1/s} \left(\frac{Bw - y}{q} \right) = 2\pi i \mathbb{E}_{X \sim D_{\mathcal{L}^*, s}} \left[X \omega_q^{\langle w, [X]_q \rangle} e^{-2\pi i \langle X, y \rangle / q} \right]. \end{aligned}$$

These functions are well defined on $\mathbb{F}_q^n$ because $F_{1/s}$ and its gradient are $\mathcal{L}$ -periodic. The next lemma identifies the vector that we estimate.

Lemma 5.2. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1+1/n)\lambda$, let $v \in \mathcal{L}$ be closest to y , let $e = v - y$, and let $w = B^{-1}v \bmod q$. Let $t_0 < t < 1/4$. Then, it holds that*

$$f_y(w)^p = 2^{-(t+o(1))n\|e\|^2/d^2} (1 + 2^{-\Omega(n)}), \quad \left\| -\frac{G_y(w)}{f_y(w)} - \frac{2\pi s^2}{q} e \right\| \leq 2^{-\Omega(n)} s^2 \lambda.$$

In particular, if $\|e\| \leq \alpha\lambda$, then $f_y(w)^p \geq 2^{-(t\alpha^2+o(1))n}$.

Proof. Since $Bw - v \in q\mathcal{L}$ and $F_{1/s}$ is $\mathcal{L}$ -periodic, the primal representations of $f_y(w)$ and $G_y(w)$ are

$$f_y(w) = \frac{1}{\rho_{1/s}(\mathcal{L})} \sum_{x \in \mathcal{L}} \rho_{1/s} \left(x + \frac{e}{q} \right), \quad G_y(w) = -\frac{2\pi s^2}{\rho_{1/s}(\mathcal{L})} \sum_{x \in \mathcal{L}} \left(x + \frac{e}{q} \right) \rho_{1/s} \left(x + \frac{e}{q} \right). \quad (73)$$

The choice of v gives $\|x + e\| \geq \|e\|$ for every $x \in \mathcal{L}$. Consequently,

$$\|x + e/q\|^2 - \|e/q\|^2 = \|x\|^2 + \frac{2}{q} \langle x, e \rangle \geq \left(1 - \frac{1}{q} \right) \|x\|^2. \quad (74)$$

Since $\rho_{1/s}(z) = \exp(-\pi s^2 \|z\|^2)$, Eq. (74) implies, for $x \neq 0$,

$$\frac{\rho_{1/s}(x + e/q)}{\rho_{1/s}(e/q)} \leq \rho_{1/(s\sqrt{1-1/q})}(x) \leq \rho_{\sqrt{2}/s}(x)$$

where we use $q \geq 2$ in the last inequality. Applying Lemma 3.3 with $k = 0$ and width $\sqrt{2}/s$ gives

$$\sum_{x \in \mathcal{L} \setminus \{0\}} \rho_{1/s}(x + e/q) \leq \rho_{1/s}(e/q) \sum_{x \in \mathcal{L} \setminus \{0\}} \rho_{\sqrt{2}/s}(x) \leq 2^{-\Omega(n)} \rho_{1/s}(e/q). \quad (75)$$

It follows from Lemma 3.1 and Eqs. (73) and (75) that

$$f_y(w) = \rho_{1/s}(e/q) (1 + 2^{-\Omega(n)}).$$

Observe that, using $s^2 = 4nt \ln 2 / (\pi d^2)$ and $q^2 = (4 + o(1))p$,

$$\rho_{1/s}(e/q)^p = \exp\left(-\frac{\pi s^2 p}{q^2} \|e\|^2\right) = 2^{-(t+o(1))n\|e\|^2/d^2}.$$

Since $p = 2^{o(n)}$ so that $(1 + 2^{-\Omega(n)})^p = 1 + 2^{-\Omega(n)}$, we obtain the first bound

$$f_y(w)^p = 2^{-(t+o(1))n\|e\|^2/d^2} (1 + 2^{-\Omega(n)}).$$

If $\|e\| \leq \alpha\lambda$, then $\|e\|^2/d^2 \leq \alpha^2$, and the last assertion follows.

We next bound $G_y(w)$. [Eq. \(73\)](#) gives

$$\frac{G_y(w)}{f_y(w)} + \frac{2\pi s^2}{q} e = -2\pi s^2 \frac{\sum_{x \in \mathcal{L}} x \rho_{1/s}(x + e/q)}{\sum_{x \in \mathcal{L}} \rho_{1/s}(x + e/q)}. \quad (76)$$

We give a bound on this expression. Since v is closest to y and $e = v - y$, for every $x \in \mathcal{L}$ we have

$$\|e\| \leq \|e + x\|, \quad \text{and hence} \quad 2\langle x, e \rangle \geq -\|x\|^2.$$

It follows that

$$\|x + e/q\|^2 - \|e/q\|^2 = \|x\|^2 + \frac{2}{q} \langle x, e \rangle \geq \left(1 - \frac{1}{q}\right) \|x\|^2 \geq \frac{1}{2} \|x\|^2.$$

Therefore, for every $x \in \mathcal{L}$,

$$\frac{\rho_{1/s}(x + e/q)}{\rho_{1/s}(e/q)} = \exp\left(-\pi s^2 \left(\|x + e/q\|^2 - \|e/q\|^2\right)\right) \leq \exp\left(-\frac{\pi s^2}{2} \|x\|^2\right) = \rho_{\sqrt{2}/s}(x).$$

Since $\|x\| \leq \lambda(1 + \|x\|/\lambda)$, applying [Lemma 3.3](#) with $k = 1$ and width $\sqrt{2}/s$ gives

$$\sum_{x \in \mathcal{L} \setminus \{0\}} \|x\| \rho_{1/s}(x + e/q) \leq \lambda \rho_{1/s}(e/q) \sum_{x \in \mathcal{L} \setminus \{0\}} \left(1 + \frac{\|x\|}{\lambda}\right) \rho_{\sqrt{2}/s}(x) \leq 2^{-\Omega(n)} \lambda \rho_{1/s}(e/q). \quad (77)$$

The denominator of [Eq. \(76\)](#) is at least $\rho_{1/s}(e/q)$, hence [Eq. \(77\)](#) gives

$$\left\| \frac{G_y(w)}{f_y(w)} + \frac{2\pi s^2}{q} e \right\| \leq 2^{-\Omega(n)} s^2 \lambda.$$

This proves the second bound. $\square$

The length of e is not known in advance. If $\|e\| < n^{-1/3}\lambda$, then y already satisfies the BDD promise. Otherwise, we can guess its length among polynomially many candidates. The next lemma formalizes this idea, together with the role of [Lemma 4.2](#) in the CVP case.

Lemma 5.3. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1+1/n)\lambda$, let $v \in \mathcal{L}$ be closest to y , let $e = v - y$, and let $w = B^{-1}v \bmod q$. Let $t_0 < t < \min\{1/4, 1/(4\alpha^2)\}$. Assume that $\|e\| \leq \alpha\lambda$. Suppose that $A \in \mathbb{C}^n$ satisfies*

$$\|A - cG_y(w)f_y(w)^{p-1}\| \leq \frac{s^2\lambda}{q} 2^{-n/4+o(n)} \quad (78)$$

for some $c \in \mathbb{C}$ with $|c| = 1$. If $\|e\| \geq n^{-1/3}\lambda$, one of the well-defined nonzero vectors $\text{Re}A/\|\text{Re}A\|$ and $\text{Im}A/\|\text{Im}A\|$ is within distance $2^{-\Omega(n)}$ of either $e/\|e\|$ or $-e/\|e\|$. For the corresponding unit vector u (i.e., one of $\text{Re}A/\|\text{Re}A\|$ or $\text{Im}A/\|\text{Im}A\|$), the following holds for all sufficiently large n : There is $j \in [\lceil 4\alpha n^{1/3} \rceil]$ such that

$$\left\| y \pm \frac{jd}{4n^{1/3}} u - v \right\| < n^{-1/3}\lambda.$$

Otherwise, if $\|e\| < n^{-1/3}\lambda$, the target y itself satisfies the BDD promise.

Proof. By Lemma 5.2, there is $e_0 \in \mathbb{R}^n$ with $\|e_0\| \leq 2^{-\Omega(n)}\lambda$ such that

$$-G_y(w)f_y(w)^{p-1} = \frac{2\pi s^2}{q}f_y(w)^p(e + e_0). \quad (79)$$

If $\|e\| \geq n^{-1/3}\lambda$, then the last statement of Lemma 5.2, $f_y(w)^p \geq 2^{-(t\alpha^2+o(1))n}$, gives

$$\|G_y(w)f_y(w)^{p-1}\| = \left\| \frac{2\pi s^2}{q}f_y(w)^p(e + e_0) \right\| \geq \frac{s^2 n^{-1/3}\lambda}{q} 2^{-(t\alpha^2+o(1))n}.$$

The error in Eq. (78) is exponentially smaller because $t\alpha^2 < 1/4$. Since the vector in Eq. (79) is real and at least one of $|\text{Rec}|$ and $|\text{Imc}|$ is at least $1/\sqrt{2}$, Lemma 2.7 gives the claimed direction as in the proof of Lemma 4.2.

Let $r = \|e\|$. Since $r \leq \alpha\lambda \leq \alpha d$, some $j \in \{0, \dots, \lceil 4\alpha n^{1/3} \rceil\}$ satisfies $|\frac{jd}{4n^{1/3}} - r| \leq \frac{d}{8n^{1/3}}$ so that $\frac{jd}{4n^{1/3}}u$ approximates e in norm. In the current case, $\|e\| = r \geq n^{-1/3}\lambda > \frac{d}{8n^{1/3}}$ for all sufficiently large n , so $j \geq 1$. For the sign satisfying $\|\pm u - e/r\| \leq 2^{-\Omega(n)}$, we have

$$\begin{aligned} \left\| y \pm \frac{jd}{4n^{1/3}}u - v \right\| &= \left\| -e \pm ru \mp ru \pm \frac{jd}{4n^{1/3}}u \right\| \leq r\|\pm u - e/r\| + \left| \frac{jd}{4n^{1/3}} - r \right| \\ &\leq \alpha\lambda 2^{-\Omega(n)} + \frac{d}{8n^{1/3}} < n^{-1/3}\lambda \end{aligned}$$

holds for sufficiently large n . This proves the case for $\|e\| \geq n^{-1/3}\lambda$. The last assertion for the other case is the definition of the $n^{-1/3}$ -BDD promise. $\square$

5.2 The coordinate decomposition and the estimator

Observe that for any $\varepsilon \in \{-1, 1\}$,

$$\mathbb{E}_X \left[\omega_q^{\langle \varepsilon w, [X]_q \rangle} e^{-2\pi i \varepsilon \langle X, y \rangle / q} \right] = F_{1/s} \left(\varepsilon \frac{Bw - y}{q} \right) = f_y(w),$$

where the last equality follows because $F_{1/s}$ is even. For independent $X_0, \dots, X_{p-1} \sim D_{\mathcal{L}^*, s}$ and for

$$\Phi_y(X_0, \dots, X_{p-1}) := \exp \left(-\frac{2\pi i}{q} \left\langle \sum_{r=0}^{p-1} \varepsilon_r X_r, y \right\rangle \right),$$

we have the following equation because of the independence and the symmetry of the discrete Gaussian

$$G_y(w)f_y(w)^{p-1} = \mathbb{E} \left[2\pi i X_0 \Phi_y(X_0, \dots, X_{p-1}) \omega_q^{\langle w, \sum_{r=0}^{p-1} \varepsilon_r [X_r]_q \rangle} \right]$$

where we use $[X_r]_q = B^T X_r \bmod q$ and $w = B^{-1}v \bmod q$.

What to estimate. Recall the notations for the spaces H_j

$$H_j := \{x = (x_1, \dots, x_n) \in \mathbb{F}_q^n : x_k = 0 \text{ for every } k \notin \{(j-1)n/m + 1, \dots, jn/m\}\},$$

with the assumption that $m = \log_2 p + 1$ divides n and the decomposition

$$\mathbb{F}_q^n = H_1 \oplus \dots \oplus H_m, \quad Y = \sum_{j=1}^m (Y)_{H_j} \quad \text{for } Y \in \mathbb{F}_q^n$$

where $(Y)_{H_j} \in H_j$ denotes the unique component satisfying $Y = \sum_{j=1}^m (Y)_{H_j}$.

Consider independent discrete Gaussian samples $X_0, \dots, X_{p-1} \sim D_{\mathcal{L}^*, s}$. Let U be uniformly random in $\text{GL}_n(\mathbb{F}_q)$. Recall the recursive definition $Y_{0,r} := U[X_r]_q = U(B^T X_r \bmod q)$ for $r \in [0, p)$ and define $Y_{j,a} := Y_{j-1,2a} - Y_{j-1,2a+1}$ for $j \in [m-1]$ and $a \in [0, p/2^j)$. For $\varepsilon_r = (-1)^{\text{wt}_2(r)}$, we have

$$Y_{m-1,0} = \sum_{r=0}^{p-1} \varepsilon_r Y_{0,r}, \quad Y_{j,a} = \sum_{b=0}^{2^j-1} \varepsilon_b Y_{0,a2^j+b} = \varepsilon_{a2^j} \sum_{r=a2^j}^{(a+1)2^j-1} \varepsilon_r Y_{0,r}.$$

Recall the constraint function from Eq. (25):

$$Q^{p-1} C_{\delta}(X_0, \dots, X_{p-1}) = Q^{p-1} \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \mathbf{1}_{\{(Y_{j,a})_{H_j} = \delta_{j,a}\}} = \sum_{\alpha} \left(\prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{-\langle \alpha_{j,a}, \delta_{j,a} \rangle} \right) \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{\langle \alpha_{j,a}, Y_{j,a} \rangle}.$$

For $\theta \in H_m$, analogously to Eq. (27), we define

$$A_{y,\delta}(\theta) := Q^{p-1} \mathbb{E} \left[C_{\delta}(X_0, \dots, X_{p-1}) 2\pi i X_0 \Phi_y(X_0, \dots, X_{p-1}) \omega_q^{\langle \theta, (Y_{m-1,0})_{H_m} \rangle} \mid U, \delta \right]. \quad (80)$$

The expectation is over the independent samples, while U and δ are fixed.

The following lemma is a CVP variant of Lemma 4.3. The proof is analogous.

Lemma 5.4. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$, let $v \in \mathcal{L}$ be closest to y , suppose that $\|v - y\| \leq \alpha\lambda$, let $w = B^{-1}v \bmod q$ be nonzero, and let $\theta_* = (U^{-T}w)_{H_m}$. Except with probability $o(1)$ over U and δ , there is $c_{\delta} \in \mathbb{C}$ with $|c_{\delta}| = 1$ such that*

$$\|A_{y,\delta}(\theta_*) - c_{\delta} G_y(w) f_y(w)^{p-1}\| \leq \frac{s^2 \lambda}{q} 2^{-n/4+o(n)}.$$

Proof. For $\alpha = (\alpha_{j,a})_{j,a}$, let $\chi_{\alpha}(\delta) := \prod_{j=1}^{m-1} \prod_{a=0}^{p/2^j-1} \omega_q^{-\langle \alpha_{j,a}, \delta_{j,a} \rangle}$. Substituting $Q^{p-1} C_{\delta}$ in Eq. (80) to the summand for α (Eq. (26)), using Eq. (28) gives

$$A_{y,\delta}(\theta) = \sum_{\alpha} \chi_{\alpha}(\delta) G_y(U^T k_{\alpha, \theta, 0}) \prod_{r=1}^{p-1} f_y(\varepsilon_r U^T k_{\alpha, \theta, r}). \quad (81)$$

Here, we use the fact that for every $k \in \mathbb{F}_q^n$ and $\varepsilon \in \{-1, 1\}$,

$$\mathbb{E}_X \left[\omega_q^{\langle U^T k, [X]_q \rangle} e^{-2\pi i \varepsilon \langle X, y \rangle / q} \right] = F_{1/s} \left(\frac{BU^T k - \varepsilon y}{q} \right) = f_y(\varepsilon U^T k).$$

Let $k_* = U^{-T}w$ and let α^* be defined by Eq. (22). Then $k_{\alpha^*, \theta_*, r} = \varepsilon_r k_*$ for every r . Hence, the summand indexed by α^* in Eq. (81) is

$$c_{\delta} G_y(w) f_y(w)^{p-1}, \quad c_{\delta} := \chi_{\alpha^*}(\delta),$$

and $|c_{\delta}| = 1$. Applying Lemma 4.4 to the remaining summands and then Lemma 5.5 below gives

$$\mathbb{E}_{\delta} \left[\|A_{y,\delta}(\theta_*) - c_{\delta} G_y(w) f_y(w)^{p-1}\|^2 \mid U \right] \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-n/2+o(n)}.$$

Markov's inequality proves the lemma. $\square$

The proof of the following estimate is deferred to Section 5.5.

Lemma 5.5. *Under the assumptions of Lemma 5.4, let $k_* = U^{-T}w$ and let α^* be defined by Eq. (22). Except with probability $o(1)$ over U ,*

$$\sum_{\alpha \neq \alpha^*} \left\| G_y(U^T k_{\alpha, \theta_*, 0}) \prod_{r=1}^{p-1} f_y(\varepsilon_r U^T k_{\alpha, \theta_*, r}) \right\|^2 \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{-n/2+o(n)}. \quad (82)$$

Estimator. We now explain how to estimate the value $A_{y,\delta}(\theta)$. For the independent samples $X_{r,i}$ and the indexed recursive differences $Y_{j,a}(\mathbf{i})$ defined in Eq. (32), we define

$$\Phi_y(\mathbf{i}) := \exp \left(-\frac{2\pi i}{q} \left\langle \sum_{r=0}^{p-1} \varepsilon_r X_{r,i_r}, y \right\rangle \right), \quad \mathbf{i} = (i_0, \dots, i_{p-1}) \in [N]^p. \quad (83)$$

For $\theta \in H_m$, our estimator is defined as follows:

$$\hat{A}_{y,\delta}(\theta) := \frac{Q^{p-1}}{N^p} \sum_{\mathbf{i} \in [N]^p} C_\delta(\mathbf{i}) 2\pi i X_{0,i_0} \Phi_y(\mathbf{i}) \omega_q^{\langle \theta, (Y_{m-1,0}(\mathbf{i}))_{H_m} \rangle}. \quad (84)$$

Lemma 5.6. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $2^{-\Omega(n)}$ over U , simultaneously for every $y \in \mathbb{R}^n$ and for every $\theta \in H_m$, it holds that*

$$\mathbb{E}_{\delta,X} \left[\left\| \hat{A}_{y,\delta}(\theta) - A_{y,\delta}(\theta) \right\|^2 \mid U \right] \leq \left(\frac{s^2 \lambda}{q} \right)^2 Q^{-1} 2^{o(n)} = \left(\frac{s^2 \lambda}{q} \right)^2 2^{-n/2+o(n)} \quad (85)$$

The constants and the exceptional probability are uniform in y .

The proof of Lemma 5.6 is also deferred to Section 5.5.

Lemma 5.7. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$, let $v \in \mathcal{L}$ be closest to y , suppose that $\|v - y\| \leq \alpha\lambda$, let $w = B^{-1}v \bmod q$ be nonzero, and let $\theta_* = (U^{-T}w)_{H_m}$. Except with probability $o(1)$ over U , δ , and the samples, there is $c_\delta \in \mathbb{C}$ with $|c_\delta| = 1$ such that*

$$\left\| \hat{A}_{y,\delta}(\theta_*) - c_\delta G_y(w) f_y(w)^{p-1} \right\| \leq \frac{s^2 \lambda}{q} 2^{-n/4+o(n)}. \quad (86)$$

Proof. With probability $1 - o(1)$, the conclusions of both Lemmas 5.4 and 5.6 hold. Applying Markov's inequality to Eq. (85) gives that

$$\left\| \hat{A}_{y,\delta}(\theta_*) - A_{y,\delta}(\theta_*) \right\| \leq \frac{s^2 \lambda}{q} 2^{-n/4+o(n)}$$

holds except with probability $o(1)$. Combining this estimate with Lemma 5.4 and applying the triangle inequality proves the lemma. $\square$

We next give the batch computation of the estimator for all θ in Algorithm 3 similar to Lemma 4.8.

Lemma 5.8. *Except with probability $o(1)$ over δ , approximations to all vectors $\hat{A}_{y,\delta}(\theta)$ for $\theta \in H_m$, each with additive error at most $(s^2 d/q) 2^{-n/4-\Omega(n)}$, can be computed in time and space $2^{n/2+o(n)}$.*

Proof. Induction on j shows that the third component of an element in $\mathbb{L}_{j,a}$ is

$$\phi = \exp \left(-\frac{2\pi i}{q} \left\langle \sum_{r=a2^j}^{(a+1)2^j-1} \varepsilon_{r-a2^j} X_{r,i_r}, y \right\rangle \right).$$

If $a = 0$, its fourth component is $Z = 2\pi i X_{0,i_0} \phi$, whereas if $a \neq 0$, its fourth component is $Z = 0$. In particular, every element chosen from $\mathbb{L}_{j-1,2a+1}$ has fourth component $Z_R = 0$, which explains the update $Z_L \overline{\phi_R}$ in Step 2(b). At the root, where $j = m - 1$ and necessarily $a = 0$, the value ϕ coincides with $\Phi_y(\mathbf{i})$ in Eq. (83). Thus, Steps 3 and 4 compute $\hat{A}_{y,\delta}(\theta)$ as defined in Eq. (84).

The difference conditions and the abort bounds are exactly those in Algorithm 1. The additional phase arithmetic incurs only polynomial overhead under the finite-precision convention in Section 2. Hence, Eqs. (36) and (37) give the claimed abort probability, time, and space. $\square$

Algorithm 3: Computation of $\widehat{A}_{y,\delta}$

1. For every $r \in [0, p)$ and $i \in [N]$, let $\phi_{r,i} := e^{-2\pi i \langle X_{r,i}, y \rangle / q}$, let $Z_{0,i} := 2\pi i X_{0,i} \phi_{0,i}$, and let $Z_{r,i} := 0$ for $r \neq 0$. Construct lists

$$\mathbf{L}_{0,r} := \{(i, Y_{r,i}, \phi_{r,i}, Z_{r,i}) : i \in [N]\} \quad \text{for } r \in [0, p).$$

2. For $j = 1, \dots, m-1$ and $a \in [0, p/2^j)$, we do:

- (a) For each $z \in H_j$, construct $\mathbf{L}_{j-1,2a+1,z} := \{(i_R, Y_R, \phi_R, Z_R) \in \mathbf{L}_{j-1,2a+1} : (Y_R)_{H_j} = z\}$.
- (b) For every $(i_L, Y_L, \phi_L, Z_L) \in \mathbf{L}_{j-1,2a}$, enumerate all elements (i_R, Y_R, ϕ_R, Z_R) in $\mathbf{L}_{j-1,2a+1, (Y_L)_{H_j} - \delta_{j,a}}$ and include $((i_L, i_R), Y_L - Y_R, \phi_L \overline{\phi_R}, Z_L \overline{Z_R})$ in $\mathbf{L}_{j,a}$. Abort immediately if $|\mathbf{L}_{j,a}| > Qp^{7 \cdot 2^j}$.

3. For every $b \in H_m$, compute $T_{y,\delta}(b) := \sum_{\substack{(i,Y,\phi,Z) \in \mathbf{L}_{m-1,0} \\ (Y)_{H_m} = b}} Z$.

4. Using a vector-valued fast Fourier transform over H_m , compute for every $\theta \in H_m$

$$\widehat{A}_{y,\delta}(\theta) = \frac{Q^{p-1}}{N^p} \sum_{b \in H_m} T_{y,\delta}(b) \omega_q^{\langle \theta, b \rangle}.$$

5.3 The algorithm

Parameters. Fix constants $\alpha > 0$ and t satisfying $t_0 < t < \min\{1/4, 1/(4\alpha^2)\}$. Use the parameters fixed at the beginning of this section, and set $N := Qp^4$. Then $p \log p = o(n)$ and $Q, N = 2^{n/2+o(n)}$.

Input. A basis $B \in \mathbb{R}^{n \times n}$ of a full-rank lattice $\mathcal{L} = B\mathbb{Z}^n$ and a target $y \in \mathbb{R}^n$ satisfying $\text{dist}(y, \mathcal{L}) \leq \alpha \lambda_1(\mathcal{L})$.

Output. A vector $v \in \mathcal{L}$ minimizing $\|v - y\|$, with probability at least $2/3$.

Algorithm 4: CVP with a distance guarantee

1. Use LLL to construct the polynomial-size set $\mathcal{D}$ of values d used in [Algorithm 2](#).
2. For every $d \in \mathcal{D}$, independently repeat the following procedure a constant number of times:
 - (a) Run the preprocessing algorithm in [Theorem 2.6](#).
 - (b) Choose a uniformly from $\mathbb{F}_q^n$, using its representative in $\{0, \dots, q-1\}^n$, and let $\tilde{y} := y + Ba$. Query the BDD algorithm at $\tilde{y}$.
 - (c) Let $s^2 = 4nt \ln 2 / (\pi d^2)$ and obtain the pN samples. Abort this repetition if a sample has norm larger than $Cs\sqrt{n}$, where C is a sufficiently large constant. Choose U and δ randomly, and run [Algorithm 3](#) for $\tilde{y}$. Abort this repetition if that subroutine aborts.
 - (d) Let $h = d/(4n^{1/3})$ and $J = \lceil 4\alpha n^{1/3} \rceil$. For every $\theta \in H_m$, every nonzero $V \in \{\text{Re} \widehat{A}_{\tilde{y},\delta}(\theta), \text{Im} \widehat{A}_{\tilde{y},\delta}(\theta)\}$, and every $j \in [J]$, query the BDD algorithm at $\tilde{y} + jhV/\|V\|$ and $\tilde{y} - jhV/\|V\|$.
 - (e) Discard every failure symbol and every returned vector outside $\mathcal{L}$. Subtract Ba from every remaining vector and collect the resulting vectors.
3. If no vector is collected, declare failure. Otherwise, return the vector minimizing its distance to y .

Proof of [Theorem 5.1](#). Fix a closest vector v to y , let $\lambda = \lambda_1(\mathcal{L})$, and let $e = v - y$. Consider a value $d \in \mathcal{D}$ satisfying $\lambda \leq d \leq (1 + 1/n)\lambda$, and fix one repetition corresponding to this value.

After the random translation $\tilde{y} = y + Ba$, $\tilde{v} := v + Ba$ is closest to $\tilde{y}$, $\tilde{v} - \tilde{y} = e$. If $\|e\| < n^{-1/3}\lambda$, the query at $\tilde{y}$ returns $\tilde{v}$ whenever the BDD preprocessing succeeds. We consider the other case in the analysis

below. Note that given random a ,

$$w := B^{-1}\tilde{v} \bmod q = B^{-1}v + a \bmod q$$

is uniform in $\mathbb{F}_q^n$. In particular, for the independently chosen U ,

$$\Pr[w \in U^T(H_1 \oplus \cdots \oplus H_{m-1})] = Q^{-1} = o(1).$$

We condition on the complementary event. Then $w \neq 0$, so that we can apply [Lemmas 5.4](#) and [5.7](#). The excluded event contributes only $o(1)$ to the failure probability.

By [Lemma 3.1](#) and [Theorem 2.2](#), the generated discrete Gaussian samples have inverse-exponentially small statistical distance from the independent samples, which only changes the probabilities negligibly small. Let $\theta_* = (U^{-T}w)_{H_m}$. By [Lemma 5.7](#), except with probability $o(1)$, the vector $\hat{A}_{\tilde{y}, \delta}(\theta_*)$ satisfies [Eq. \(78\)](#) for some $c \in \mathbb{C}$ with $|c| = 1$. Applying [Lemma 5.3](#), there are a nonzero vector $V \in \{\operatorname{Re}\hat{A}_{\tilde{y}, \delta}(\theta_*), \operatorname{Im}\hat{A}_{\tilde{y}, \delta}(\theta_*)\}$, an index $j \in [J]$, and a sign $\sigma \in \{-1, 1\}$ such that

$$\left\| \tilde{y} + \sigma j h \frac{V}{\|V\|} - \tilde{v} \right\| < n^{-1/3} \lambda.$$

The algorithm queries the BDD algorithm at this point, since it considers every such V , j , and sign. Hence, successful BDD preprocessing returns $\tilde{v}$. After subtracting Ba , the algorithm therefore obtains v , which will be returned (or one among the same minimal distance).

The sample generation and [Lemma 5.8](#) use time and space $2^{n/2+o(n)}$ in each repetition. The shifted-query step makes $Q2^{o(n)}$ BDD queries because $J = O(n^{1/3})$, so its total time has the same bound. The $n^2 + 1$ choices of d and the constant number of repetitions incur only a polynomial factor.

Each repetition for the target value d succeeds with constant probability by [Lemmas 3.2](#), [5.7](#) and [5.8](#) and [Theorem 2.6](#). A constant number of repetitions raises the success probability to at least $2/3$. $\square$

5.4 Random target CVP in a random lattice

Let μ_n denote the Haar probability measure on $X_n := \operatorname{SL}_n(\mathbb{R})/\operatorname{SL}_n(\mathbb{Z})$, the space of covolume-one lattices in $\mathbb{R}^n$. In this subsection, $\mathcal{L} \sim \mu_n$ and, conditional on $\mathcal{L}$, the target $y \bmod \mathcal{L}$ is uniform in $\mathbb{R}^n/\mathcal{L}$. We have the following result.

Corollary 5.9. *Let $\mathcal{L} \sim \mu_n$. With probability $1 - \exp(-\Omega(n^{2/3}))$ over $\mathcal{L}$, [Algorithm 4](#) solves a uniformly random CVP target $(y, \mathcal{L})$ with probability at least $1/2$ for all sufficiently large n over $y \bmod \mathcal{L}$ and the internal randomness of the algorithm.*

The corollary follows from the following lemma because, for $\alpha = 1.01 < 1/2\sqrt{t_0} = 1.039 \cdots$ and all sufficiently large n , the bound in the lemma below implies $\operatorname{dist}(y, \mathcal{L}) \leq \alpha \lambda_1(\mathcal{L})$, under which one execution of [Algorithm 4](#) succeeds with probability at least $2/3$.

Lemma 5.10. *With probability $1 - \exp(-\Omega(n^{2/3}))$ over $\mathcal{L} \sim \mu_n$, we have*

$$\Pr_{y \bmod \mathcal{L}} \left[\operatorname{dist}(y, \mathcal{L}) \leq \frac{1 + n^{-1/3}}{1 - n^{-1/3}} \lambda_1(\mathcal{L}) \right] \geq 1 - \exp(-\Omega(n^{2/3})). \quad (87)$$

Proof. Let V_n be the volume of the Euclidean unit ball, let $r_n = V_n^{-1/n}$. For $R > 0$, let $N_R(\mathcal{L})$ be the number of nonzero vectors of $\mathcal{L}$ in the Euclidean ball of radius R . Siegel's mean-value formula [\[49\]](#) gives $\mathbb{E}[N_R(\mathcal{L})] = V_n R^n$, and hence

$$\Pr_{\mathcal{L}} \left[\lambda_1(\mathcal{L}) < (1 - n^{-1/3}) r_n \right] \leq (1 - n^{-1/3})^n. \quad (88)$$

Let $M_R(\mathcal{L}, y)$ be the number of $x \in \mathcal{L}$ satisfying $\|x - y\| \leq R$, and write $V = V_n R^n$. Let $\mathcal{F}$ be a fundamental domain of $\mathcal{L}$ and let $\mathbf{1}_{B_R}$ denote the indicator of the Euclidean ball B_R of radius R . Since $\mathcal{L}$

has covolume one, $\text{vol}(\mathcal{F}) = 1$, and $M_R(\mathcal{L}, y) = \sum_{x \in \mathcal{L}} \mathbf{1}_{B_R}(x - y)$. Unfolding the integral over $\mathcal{F}$ gives, for every fixed $\mathcal{L}$,

$$\mathbb{E}_{y \bmod \mathcal{L}}[M_R(\mathcal{L}, y)] = \sum_{x \in \mathcal{L}} \int_{\mathcal{F}} \mathbf{1}_{B_R}(x - y) dy = \int_{\mathbb{R}^n} \mathbf{1}_{B_R}(u) du = V.$$

Similarly, expanding the square and writing $z = x - x'$ gives

$$\begin{aligned} \mathbb{E}_{y \bmod \mathcal{L}}[M_R(\mathcal{L}, y)^2] &= \sum_{x, x' \in \mathcal{L}} \int_{\mathcal{F}} \mathbf{1}_{B_R}(x - y) \mathbf{1}_{B_R}(x' - y) dy \\ &= \sum_{z \in \mathcal{L}} \sum_{x \in \mathcal{L}} \int_{\mathcal{F}} \mathbf{1}_{B_R}(x - y) \mathbf{1}_{B_R}(x - z - y) dy \\ &= \sum_{z \in \mathcal{L}} \int_{\mathbb{R}^n} \mathbf{1}_{B_R}(u) \mathbf{1}_{B_R}(u - z) du \\ &= \sum_{z \in \mathcal{L}} \text{vol}(B_R \cap (B_R + z)). \end{aligned}$$

Let $h(z) := \text{vol}(B_R \cap (B_R + z))$. The term corresponding to $z = 0$ is $h(0) = \text{vol}(B_R) = V$. Siegel's mean-value formula applied to h gives

$$\mathbb{E}_{\mathcal{L} \sim \mu_n} \left[\sum_{z \in \mathcal{L} \setminus \{0\}} h(z) \right] = \int_{\mathbb{R}^n} h(z) dz.$$

Moreover, by Fubini's theorem and translation invariance of Lebesgue measure,

$$\begin{aligned} \int_{\mathbb{R}^n} h(z) dz &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{1}_{B_R}(u) \mathbf{1}_{B_R}(u - z) du dz \\ &= \int_{\mathbb{R}^n} \mathbf{1}_{B_R}(u) \left(\int_{\mathbb{R}^n} \mathbf{1}_{B_R}(u - z) dz \right) du = V^2. \end{aligned}$$

Consequently,

$$\mathbb{E}_{\mathcal{L}, y}[M_R(\mathcal{L}, y)^2] = V^2 + V, \quad \text{Var}_{\mathcal{L}, y}(M_R(\mathcal{L}, y)) = V.$$

Chebyshev's inequality gives

$$\Pr_{\mathcal{L}, y}[\text{dist}(y, \mathcal{L}) > R] = \Pr_{\mathcal{L}, y}[M_R(\mathcal{L}, y) = 0] \leq V^{-1}. \quad (89)$$

Taking $R = (1 + n^{-1/3})r_n$ and combining [Eqs. \(88\)](#) and [\(89\)](#) gives

$$\Pr_{\mathcal{L}, y} \left[\frac{\text{dist}(y, \mathcal{L})}{\lambda_1(\mathcal{L})} > \frac{1 + n^{-1/3}}{1 - n^{-1/3}} \right] \leq (1 - n^{-1/3})^n + (1 + n^{-1/3})^{-n} = \exp(-\Omega(n^{2/3})). \quad (90)$$

Since $\frac{1+n^{-1/3}}{1-n^{-1/3}} = 1 + O(n^{-1/3})$, this proves the claimed bound jointly over $\mathcal{L}$ and $y \bmod \mathcal{L}$.

Finally, let $b(\mathcal{L})$ be the conditional probability over $y \bmod \mathcal{L}$ that [Eq. \(87\)](#) fails, or more precisely,

$$b(\mathcal{L}) := \Pr_{y \bmod \mathcal{L}} \left[\frac{\text{dist}(y, \mathcal{L})}{\lambda_1(\mathcal{L})} > \frac{1 + n^{-1/3}}{1 - n^{-1/3}} \right].$$

By [Eq. \(90\)](#), there is a constant $c > 0$ such that, for all sufficiently large n ,

$$\mathbb{E}_{\mathcal{L}}[b(\mathcal{L})] \leq \exp(-cn^{2/3}).$$

Markov's inequality therefore gives

$$\Pr_{\mathcal{L}}[b(\mathcal{L}) > \exp(-cn^{2/3}/2)] \leq \exp(-cn^{2/3}/2).$$

This proves the lemma. □

5.5 Proofs of the CVP lemmas

We use the notation from [Section 4.4](#) as follows:

$$\begin{aligned} P_j &= H_1 \oplus \cdots \oplus H_j, & S_j &= U^T P_j, & R_j &= H_{j+1} \oplus \cdots \oplus H_m, \\ K_j &= \{Bx : x \in \mathbb{Z}^n, x \bmod q \in S_j\}, & \sigma_j &= \frac{q}{2^{j/2}s}, \\ \mathcal{A}_{\leq j} &= \left\{ (\alpha_{h,a})_{\substack{1 \leq h \leq j \\ 0 \leq a < 2^{j-h}}} : \alpha_{h,a} \in H_h \right\}, & k_{\alpha \leq j, \kappa, r} &= \varepsilon_r \kappa + \sum_{h=1}^j \varepsilon_{r \bmod 2^h} \alpha_{h, \lfloor r/2^h \rfloor}. \end{aligned}$$

The proofs in this section follow the corresponding proofs for SVP in that subsection with small changes, which will be clearly noted below.

Recursive mass representations. For $\kappa \in \mathbb{F}_q^n$, we define the shifted versions of V and T in [Eq. \(41\)](#)

$$V_{0,y}(\kappa) := |f_y(U^T \kappa)|^2, \quad T_{0,y}(\kappa) := \|G_y(U^T \kappa)\|^2,$$

and, for $j \in \{1, \dots, m-1\}$, define recursively following [Eq. \(42\)](#)

$$V_{j,y}(\kappa) := \sum_{\alpha \in H_j} V_{j-1,y}(\kappa + \alpha)^2, \quad T_{j,y}(\kappa) := \sum_{\alpha \in H_j} T_{j-1,y}(\kappa + \alpha) V_{j-1,y}(\kappa + \alpha). \quad (91)$$

These functions are P_j -periodic. The following lemma is analogous to [Lemma 4.10](#).

Lemma 5.11. *For every $j \in \{0, \dots, m-1\}$ and $\kappa \in \mathbb{F}_q^n$,*

$$\begin{aligned} V_{j,y}(\kappa) &= \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \prod_{r=0}^{2^j-1} |f_y(\varepsilon_r U^T k_{\alpha \leq j, \kappa, r})|^2, \\ T_{j,y}(\kappa) &= \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \left\| G_y(U^T k_{\alpha \leq j, \kappa, 0}) \prod_{r=1}^{2^j-1} f_y(\varepsilon_r U^T k_{\alpha \leq j, \kappa, r}) \right\|^2. \end{aligned}$$

Proof. We follow the induction in the proof of [Lemma 4.10](#). The case $j = 0$ is obvious by definition. Decompose $\alpha \leq j$ as $(\alpha, \beta^L, \beta^R)$ using [Eq. \(45\)](#). The left half is indexed by $(\beta^L, \kappa + \alpha)$. For $0 \leq r < 2^{j-1}$, the second identity in [Eq. \(46\)](#) and $\varepsilon_{2^{j-1}+r} = -\varepsilon_r$ give

$$\varepsilon_{2^{j-1}+r} k_{\alpha \leq j, \kappa, 2^{j-1}+r} = \varepsilon_r k_{-\beta^R, \kappa + \alpha, r}. \quad (92)$$

Since $\beta^R \mapsto -\beta^R$ is a bijection of $\mathcal{A}_{\leq j-1}$, the scalar sums from both halves equal $V_{j-1,y}(\kappa + \alpha)$. This gives the first recursion in [Eq. \(91\)](#).

For the second recursion, the index $r = 0$, at which G_y appears, belongs to the left half. Hence the left half gives $T_{j-1,y}(\kappa + \alpha)$, while the right half gives $V_{j-1,y}(\kappa + \alpha)$. Summing over $\alpha \in H_j$ gives the second recursion. This proves the lemma. $\square$

Shifted Gaussian mass bounds. The next lemma is the CVP counterpart of [Lemmas 4.11](#) and [4.16](#).

Lemma 5.12. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. Except with probability $2^{-\Omega(n)}$ over U , simultaneously for every $y \in \mathbb{R}^n$, the following estimates hold. For every $j \in \{1, \dots, m-1\}$,*

$$\sum_{\kappa \in R_j} V_{j,y}(\kappa) \leq 2^{((m-j-1)/2 + o(1))n}, \quad \sum_{\kappa \in R_j} T_{j,y}(\kappa) \leq \left(\frac{s^2 \lambda}{q} \right)^2 2^{((m-j-1)/2 + o(1))n}. \quad (93)$$

Moreover, for every $j \in \{0, \dots, m-2\}$ and every $\kappa \in \mathbb{F}_q^n$,

$$\sum_{\alpha \in H_{j+1}} V_{j,y}(\kappa + \alpha) \leq 1 + 2^{-\Omega(n)}, \quad \sum_{\alpha \in H_{j+1}} T_{j,y}(\kappa + \alpha) \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{o(n)}. \quad (94)$$

Proof. By [Lemma 4.9](#), except with probability $2^{-\Omega(n)}$ over U , the Gaussian-mass estimates in [Eq. \(39\)](#) hold simultaneously for every j . Fix any U for which these estimates hold. We prove the claimed inequalities for this U , simultaneously for every $y \in \mathbb{R}^n$.

We first prove the two bounds involving $V_{j,y}$. We refer to

$$\sum_{\kappa \in R_j} V_{j,y}(\kappa) \quad \text{and} \quad \sum_{\gamma \in H_{j+1}} V_{j,y}(\kappa + \gamma)$$

as the R_j -sum and the H_{j+1} -sum, respectively.

For the R_j -sum, [Lemma 5.11](#) gives

$$\sum_{\kappa \in R_j} V_{j,y}(\kappa) = \sum_{\kappa \in R_j} \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \prod_{r=0}^{2^j-1} |f_y(z_r)|^2,$$

where $z_r := \varepsilon_r U^T k_{\alpha \leq j, \kappa, r}$. Similarly, for $w_r := \varepsilon_r U^T k_{\beta \leq j, \kappa + \gamma, r}$, the H_{j+1} -sum for fixed κ is

$$\sum_{\gamma \in H_{j+1}} V_{j,y}(\kappa + \gamma) = \sum_{\gamma \in H_{j+1}} \sum_{\beta \leq j \in \mathcal{A}_{\leq j}} \prod_{r=0}^{2^j-1} |f_y(w_r)|^2.$$

For every $\xi \in \mathbb{F}_q^n$, the primal representation of f_y gives

$$|f_y(\xi)|^2 = \frac{1}{\rho_{1/s}(\mathcal{L})^2} \sum_{x_0, x_1 \in B\xi - y + q\mathcal{L}} \exp\left(-\frac{\pi s^2}{q^2} (\|x_0\|^2 + \|x_1\|^2)\right).$$

Accordingly, expanding the R_j -sum and the H_{j+1} -sum introduces

$$u_{r,0}, u_{r,1} \in Bz_r - y + q\mathcal{L}, \quad \text{and} \quad v_{r,0}, v_{r,1} \in Bw_r - y + q\mathcal{L},$$

respectively. To deal with two cases simultaneously, we define

$$x_{r,b}^R := u_{r,b}, \quad x_{r,b}^H := v_{r,b}.$$

For each $X \in \{R, H\}$, define $a_{j,0}^X$ and $(d_{h,r}^X)_{h,r}$ by

$$\Psi_j((x_{r,b}^X)_{r,b}) = (a_{j,0}^X, (d_{h,r}^X)_{h,r}).$$

Adding y to every entry removes the common shift:

$$u_{r,b} + y \in Bz_r + q\mathcal{L}, \quad v_{r,b} + y \in Bw_r + q\mathcal{L}.$$

Moreover, the definition of the average-difference transformation gives for each $X \in \{R, H\}$

$$\Psi_j((x_{r,b}^X + y)_{r,b}) = (a_{j,0}^X + y, (d_{h,r}^X)_{h,r}).$$

Thus [Claims 4.12](#) and [4.13](#) apply to the tuples $(u_{r,b} + y)_{r,b}$, while [Claims 4.17](#) and [4.18](#) apply to the tuples $(v_{r,b} + y)_{r,b}$. We obtain the following two claims.

Claim 5.13. Fix $y \in \mathbb{R}^n$.

- In the R_j -sum, the tuple $(u_{r,b})_{r,b}$ uniquely determines $(\alpha^{\leq j}, \kappa)$.
- For fixed κ , in the H_{j+1} -sum, the tuple $(v_{r,b})_{r,b}$ uniquely determines $(\beta^{\leq j}, \gamma)$.

Consequently, the R_j -sum can be indexed by $(u_{r,b})_{r,b}$, and, for fixed κ , the H_{j+1} -sum can be indexed by $(v_{r,b})_{r,b}$, without multiplicity.

Claim 5.14. Fix $y \in \mathbb{R}^n$.

- In the R_j -sum, if $a_{j,0}^R$ is allowed to vary while all $d_{h,r}^R$ are fixed, its possible values are contained in a single coset of $\mathcal{L}$.
- In the H_{j+1} -sum, if $a_{j,0}^H$ is allowed to vary while all $d_{h,r}^H$ are fixed, its possible values are contained in a single coset of K_{j+1} .
- Every $d_{0,r}^R$ and $d_{0,r}^H$ belongs to K_0 .
- For $X \in \{R, H\}$, $h \in [j]$, and $r \in [0, 2^{j-h})$, if $d_{h,r}^X$ is allowed to vary while all lower-level differences $d_{\ell,r'}^X$ with $\ell < h$ are fixed, its possible values are contained in a single coset of K_{h-1} .

Each coset may depend on the fixed variables and on y .

By [Claim 5.13](#) and the bijectivity of Ψ_j , every term in either expansion is counted exactly once in the transformed variables. For each $X \in \{R, H\}$, [Eq. \(49\)](#) gives

$$\sum_{r=0}^{2^j-1} \sum_{b \in \{0,1\}} \|x_{r,b}^X\|^2 = 2^{j+1} \|a_{j,0}^X\|^2 + \frac{1}{2} \sum_{r=0}^{2^j-1} \|d_{0,r}^X\|^2 + \sum_{h=1}^j 2^{h-1} \sum_{r=0}^{2^{j-h}-1} \|d_{h,r}^X\|^2.$$

We first sum over $a_{j,0}^X$ and then over the variables $d_{h,r}^X$ in decreasing order of h . For $j \geq 1$, their numbers, containing cosets, and Gaussian widths are as follows:

variable	number	R_j -sum	H_{j+1} -sum	width
$a_{j,0}$	1	a coset of $\mathcal{L}$	a coset of K_{j+1}	$\frac{q}{2^{(j+1)/2}s} = \sigma_{j+1}$
$d_{0,r}$	2^j	K_0	K_0	$\frac{\sqrt{2}q}{s}$
$d_{1,r}$	2^{j-1}	a coset of K_0	a coset of K_0	$\frac{q}{s}$
$d_{h+1,r}$	2^{j-h-1}	a coset of K_h	a coset of K_h	$\frac{q}{2^{h/2}s} = \sigma_h \quad (1 \leq h < j).$

For $j = 0$, only the H_1 -sum is needed. Its variables are $a_{0,0}^H$, contained in a coset of K_1 with width σ_1 , and $d_{0,0}^H \in K_0$, with width $\sqrt{2}q/s$.

Applying [Claim 5.14](#) and [Lemma 2.1](#) at each step bounds each expansion by the product of the corresponding unshifted Gaussian masses listed above.

For the R_j -sum, this gives the right-hand side of [Eq. \(51\)](#). The estimates following that equation give

$$\sum_{\kappa \in R_j} V_{j,y}(\kappa) \leq 1 + 2^{((m-j-1)/2 - g/2 + o(1))n} \leq 2^{((m-j-1)/2 + o(1))n}.$$

For the H_{j+1} -sum, this gives the final right-hand side of [Eq. \(65\)](#) when $j = 0$ and of [Eq. \(67\)](#) when $j \geq 1$. Therefore

$$\sum_{\gamma \in H_{j+1}} V_{j,y}(\kappa + \gamma) \leq 1 + 2^{-\Omega(n)}.$$

These estimates hold simultaneously for every $y \in \mathbb{R}^n$, since the event fixed above depends only on U and [Lemma 2.1](#) holds for every shift.

It remains to prove the two inequalities involving $T_{j,y}$. The same Cauchy–Schwarz argument as in [Eq. \(53\)](#) bounds the corresponding expansion by the expansion used above for $V_{j,y}$ with the additional factor

$$\left(\frac{2\pi s^2}{q} \right)^2 \|x_{0,0}^X\|^2$$

for $X \in \{R, H\}$. Let

$$\mathcal{E}_X := \sum_{r=0}^{2^j-1} \sum_{b \in \{0,1\}} \|x_{r,b}^X\|^2.$$

Since $\|x_{0,0}^X\|^2 \leq \mathcal{E}_X$, [Eq. \(54\)](#) gives

$$\|x_{0,0}^X\|^2 e^{-\pi s^2 \mathcal{E}_X / q^2} \leq \frac{nq^2}{\pi s^2} e^{-\pi(1-1/n)s^2 \mathcal{E}_X / q^2}.$$

the preceding calculation for V 's above applies with every Gaussian width multiplied by $1/\sqrt{1-1/n}$ and with the additional factor

$$\left(\frac{2\pi s^2}{q}\right)^2 \frac{nq^2}{\pi s^2} = 4\pi n s^2 \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{o(n)},$$

where we use $s^2 \lambda^2 = \Theta(n)$ and $q = n^{O(1)}$.

The remaining argument is the same as in the proofs of [Lemmas 4.11](#) and [4.16](#), with two changes. First, the common translation by $-y$ remains only in the final average, and its Gaussian mass is bounded by the corresponding unshifted mass using [Lemma 2.1](#). Second, for the R_j -sum, we retain the zero term in this unshifted mass. Consequently, its leading Gaussian mass is

$$\rho_{\sigma_{j+1}/\sqrt{1-1/n}}(\mathcal{L}) \leq 2^{((m-j-1)/2+o(1))n},$$

whereas for the H_{j+1} -sum it is

$$\rho_{\sigma_{j+1}/\sqrt{1-1/n}}(K_{j+1}) \leq 1 + 2^{-\Omega(n)}.$$

All difference-variable masses are bounded as in the corresponding SVP proofs using [Eq. \(39\)](#) and the bounds for K_0 . Therefore,

$$\sum_{\kappa \in R_j} T_{j,y}(\kappa) \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{((m-j-1)/2+o(1))n},$$

and, including the case $j = 0$,

$$\sum_{\gamma \in H_{j+1}} T_{j,y}(\kappa + \gamma) \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{o(n)}.$$

These comparisons hold for every shift, so the bounds hold simultaneously for every $y \in \mathbb{R}^n$, completing the proof. $\square$

The nonprincipal mass. Fix $w \in \mathbb{F}_q^n \setminus \{0\}$ independently of U , let $k_* := U^{-T}w$, and, for $j \geq 1$, define

$$\Delta_{j,y} := \sum_{\alpha \in H_j \setminus \{0\}} V_{j-1,y}(k_* + \alpha)^2, \quad \Gamma_{j,y} := \sum_{\alpha \in H_j \setminus \{0\}} T_{j-1,y}(k_* + \alpha) V_{j-1,y}(k_* + \alpha).$$

Lemma 5.15. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. For fixed y and $w \neq 0$, except with probability $o(1)$ over U , simultaneously for every $j \in \{1, \dots, m-1\}$,*

$$\Delta_{j,y} \leq 2^{-n/2+o(n)}, \quad \Gamma_{j,y} \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{-n/2+o(n)}. \quad (95)$$

Proof. We follow the proof of [Lemma 4.14](#), using the total bounds in [Eq. \(93\)](#). Under the bijection $\kappa + P_{j-1} \mapsto U^T \kappa + S_{j-1}$, the terms in $\Delta_{j,y}$ and $\Gamma_{j,y}$ above with $\alpha \neq 0$ correspond to the S_{j-1} -cosets other than $w + S_{j-1}$ that are contained in $w + S_j$. Conditional on $S_0, \dots, S_{j-1}$, each fixed such coset is included with probability at most $2q^{-(n-jn/m)}$.

Let $\mathcal{G}_j$ be the event that [Eq. \(93\)](#) holds at level j . As in the SVP proof, summing [Eq. \(91\)](#) over R_j shows that the total weights assigned to the S_{j-1} -cosets are $\sum_{\kappa \in R_j} V_{j,y}(\kappa)$ and $\sum_{\kappa \in R_j} T_{j,y}(\kappa)$. These weights and $\mathcal{G}_j$ are fixed after conditioning on $S_0, \dots, S_{j-1}$. Hence linearity of conditional expectation gives

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\mathcal{G}_j} \mathbf{1}_{\{w \notin S_{m-1}\}} \Delta_{j,y} \mid S_0, \dots, S_{j-1}] &\leq 2q^{-(n-jn/m)} 2^{((m-j-1)/2+o(1))n}, \\ \mathbb{E}[\mathbf{1}_{\mathcal{G}_j} \mathbf{1}_{\{w \notin S_{m-1}\}} \Gamma_{j,y} \mid S_0, \dots, S_{j-1}] &\leq \left(\frac{s^2 \lambda}{q}\right)^2 2q^{-(n-jn/m)} 2^{((m-j-1)/2+o(1))n}.\end{aligned}$$

Since $\log_2 q = (m+1)/2 + o(1)$, the common factor on the right is

$$q^{-(n-jn/m)} 2^{((m-j-1)/2+o(1))n} = 2^{-(1/2+(m-j)/(2m)-o(1))n}.$$

Markov's inequality, with a slightly larger $o(n)$ term in the threshold, and a union bound over j prove [Eq. \(95\)](#) on $\bigcap_j \mathcal{G}_j$. Finally, $\Pr[w \in S_{m-1}] = Q^{-1}(1 + o(1)) = o(1)$, and $\Pr[\bigcap_j \mathcal{G}_j] = 1 - 2^{-\Omega(n)}$ by [Lemma 5.12](#). $\square$

Proof of [Lemma 5.5](#). Let $k_* = U^{-T}w$ and define

$$\begin{aligned}V_{j,y}^* &:= |f_y(w)|^{2^{j+1}}, & B_{j,y} &:= V_{j,y}(k_*) - V_{j,y}^*, \\ T_{j,y}^* &:= \|G_y(w)\|^2 |f_y(w)|^{2(2^j-1)}, & E_{j,y} &:= T_{j,y}(k_*) - T_{j,y}^*.\end{aligned}$$

We follow the proof of [Lemma 4.5](#). The choice $\alpha = 0$ at every level gives the displayed terms above, so $B_{0,y} = E_{0,y} = 0$. Separating that choice in [Eq. \(91\)](#) gives

$$\begin{aligned}B_{j,y} &= 2V_{j-1,y}^* B_{j-1,y} + B_{j-1,y}^2 + \Delta_{j,y}, \\ E_{j,y} &= E_{j-1,y}(V_{j-1,y}^* + B_{j-1,y}) + T_{j-1,y}^* B_{j-1,y} + \Gamma_{j,y}.\end{aligned}$$

By the primal representation and [Lemma 2.1](#), $0 < f_y(w) \leq 1$. Moreover, [Lemma 5.2](#) and $\|v - y\| \leq \alpha\lambda$ give

$$\|G_y(w)\| \leq O\left(\frac{s^2 \lambda}{q}\right) f_y(w), \quad T_{j,y}^* \leq O\left(\left(\frac{s^2 \lambda}{q}\right)^2\right).$$

Using [Lemma 5.15](#) in the two recurrences, the same induction as in the SVP proof gives

$$B_{j,y} \leq 2^{-n/2+o(n)}, \quad E_{j,y} \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{-n/2+o(n)}$$

for every $j \leq m-1$. The factor accumulated over $m = O(\log n)$ levels is absorbed into the $o(n)$ terms. Finally, $k_* - \theta_* \in P_{m-1}$, and the term obtained by choosing $\alpha = 0$ in every recurrence at k_* is the term indexed by α^* at θ_* . Hence [Lemma 5.11](#) identifies $E_{m-1,y}$ with the sum in [Eq. \(82\)](#), proving the lemma. $\square$

The second moment bound. We now introduce the CVP version of the quantities used in [Lemma 4.19](#). For $I \subseteq [0, 2^j)$, let

$$V_{j,I,y}(\kappa) := \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \prod_{r \in [0, 2^j) \setminus I} |f_y(\varepsilon_r U^T k_{\alpha \leq j, \kappa, r})|^2.$$

For $0 \notin I$, define

$$T_{j,I,y}(\kappa) := \sum_{\alpha \leq j \in \mathcal{A}_{\leq j}} \left\| G_y(U^T k_{\alpha \leq j, \kappa, 0}) \prod_{r \in [1, 2^j) \setminus I} f_y(\varepsilon_r U^T k_{\alpha \leq j, \kappa, r}) \right\|^2.$$

Thus $V_{j,\emptyset,y} = V_{j,y}$ and $T_{j,\emptyset,y} = T_{j,y}$.

Lemma 5.16. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. On the event in [Lemma 5.12](#), simultaneously for every y , $j \in \{0, \dots, m-1\}$, $\kappa \in \mathbb{F}_q^n$, and every nonempty $I \subseteq [0, 2^j)$,*

$$V_{j,I,y}(\kappa) \leq Q^{|I|-1} 2^{o(n)}, \quad T_{j,I,y}(\kappa) \leq \left(\frac{s^2 \lambda}{q}\right)^2 Q^{|I|-1} 2^{o(n)} \quad \text{if } 0 \notin I.$$

Proof. We follow the three-case induction in the proof of [Lemma 4.19](#). For $j = 0$, the only nonempty choice is $I = \{0\}$ and $V_{0,\{0\},y} = 1$. Suppose that $j \geq 1$ and let $I_L := I \cap [0, 2^{j-1})$ and $I_R := \{r \in [0, 2^{j-1}) : 2^{j-1} + r \in I\}$. The reindexing in [Eq. \(92\)](#) gives

$$\begin{aligned} V_{j,I,y}(\kappa) &= \sum_{\alpha \in H_j} V_{j-1,I_L,y}(\kappa + \alpha) V_{j-1,I_R,y}(\kappa + \alpha), \\ T_{j,I,y}(\kappa) &= \sum_{\alpha \in H_j} T_{j-1,I_L,y}(\kappa + \alpha) V_{j-1,I_R,y}(\kappa + \alpha), \end{aligned}$$

where the second identity is used only when $0 \notin I$.

If exactly one of I_L, I_R is nonempty, apply the induction hypothesis for each α to the term whose omitted set is nonempty. The other term is $V_{j-1,\emptyset,y}(\kappa + \alpha) = V_{j-1,y}(\kappa + \alpha)$, whose sum over $\alpha \in H_j$ is bounded by the first inequality in [Eq. \(94\)](#). If both I_L and I_R are nonempty, apply the induction hypothesis to both terms for each α and sum over the Q choices of α . The resulting power of Q is

$$Q^{|I_L|-1} Q^{|I_R|-1} Q = Q^{|I|-1}.$$

For the bound on $T_{j,I,y}$, the assumption $0 \notin I$ implies $0 \notin I_L$. If $I_L \neq \emptyset$ and $I_R = \emptyset$, apply the induction hypothesis to $T_{j-1,I_L,y}(\kappa + \alpha)$ for each α and use the first inequality in [Eq. \(94\)](#) to sum $V_{j-1,y}(\kappa + \alpha)$ over α . If $I_L = \emptyset$ and $I_R \neq \emptyset$, use the second inequality in [Eq. \(94\)](#) to sum $T_{j-1,y}(\kappa + \alpha)$ over α , and apply the induction hypothesis to $V_{j-1,I_R,y}(\kappa + \alpha)$ for each α . If both I_L and I_R are nonempty, apply the induction hypotheses to $T_{j-1,I_L,y}(\kappa + \alpha)$ and $V_{j-1,I_R,y}(\kappa + \alpha)$ for each α , and then sum over the Q choices of α .

For completeness, the first inequality in [Eq. \(94\)](#) is invoked at most $2^j \leq p$ times, and the resulting multiplicative contribution is

$$(1 + 2^{-\Omega(n)})^p = 1 + 2^{-\Omega(n)}.$$

At every level of the recursion for $T_{j,I,y}$, exactly one term is of the form $T_{j',I',y}$. If $I' = \emptyset$, its sum is bounded directly by the second inequality in [Eq. \(94\)](#); otherwise the induction continues through this term. Hence the second inequality is invoked at most once, and all accumulated factors are absorbed into $2^{o(n)}$. $\square$

For $\mathbf{i} \in [N]^p$ and $\theta \in H_m$, denote the corresponding summand of [Eq. \(84\)](#) by

$$\mathcal{K}_{\mathbf{i}}^y(\theta) := Q^{p-1} C_{\delta}(\mathbf{i}) 2\pi i X_{0,i_0} \Phi_y(\mathbf{i}) \omega_q^{(\theta, (Y_{m-1,0}(\mathbf{i}))_{H_m})}.$$

Then $\hat{A}_{y,\delta}(\theta) = N^{-p} \sum_{\mathbf{i}} \mathcal{K}_{\mathbf{i}}^y(\theta)$ and $\mathbb{E}_X[\mathcal{K}_{\mathbf{i}}^y(\theta) \mid U, \delta] = A_{y,\delta}(\theta)$.

Lemma 5.17. *Let $\lambda = \lambda_1(\mathcal{L}) \leq d \leq (1 + 1/n)\lambda$. On the event in [Lemma 5.12](#), simultaneously for every y , $\theta \in H_m$, and $\mathbf{i}, \mathbf{i}' \in [N]^p$, let $I := \{r \in [0, p) : i_r = i'_r\}$. If $I = \emptyset$, then*

$$\mathbb{E}_{\delta,X}[(\mathcal{K}_{\mathbf{i}}^y(\theta) - A_{y,\delta}(\theta))^*(\mathcal{K}_{\mathbf{i}'}^y(\theta) - A_{y,\delta}(\theta)) \mid U] = 0.$$

If $I \neq \emptyset$, the absolute value of this expectation is at most

$$\left(\frac{s^2 \lambda}{q}\right)^2 Q^{|I|-1} 2^{o(n)}. \tag{96}$$

Proof. We follow the proof of [Lemma 4.15](#) and spell out the target phases. If $I = \emptyset$, then, for fixed U and δ , the two collections $(X_{r,i_r})_{r \in [0,p)}$ and $(X_{r,i'_r})_{r \in [0,p)}$ are independent. Each displayed factor of the form $\mathcal{K}_i^y(\theta) - A_{y,\delta}(\theta)$ has expectation zero over its collection, so the expectation of their product is zero.

Suppose that $I \neq \emptyset$. Expand

$$Q^{p-1} C_\delta(i) \omega_q^{\langle \theta, (Y_{m-1,0}(i))_{H_m} \rangle}$$

and the analogous expression indexed by i' using [Eq. \(26\)](#). Averaging over δ and applying [Eq. \(28\)](#) and [Lemma 4.4](#) makes every term with $\alpha \neq \alpha'$ vanish. Hence only the terms with $\alpha = \alpha'$ remain.

Fix α and consider the term with $\alpha' = \alpha$. For $r \in [0, p)$, let $z_{\alpha,r} := U^T k_{\alpha,\theta,r}$ and define

$$\psi_{\alpha,r}(X) := \omega_q^{\langle z_{\alpha,r}, [X]_q \rangle} \exp\left(-\frac{2\pi i \varepsilon_r}{q} \langle X, y \rangle\right).$$

Then, it holds that

$$\mathbb{E}_X[\psi_{\alpha,r}(X)] = f_y(\varepsilon_r z_{\alpha,r}), \quad \mathbb{E}_X[2\pi i X \psi_{\alpha,0}(X)] = G_y(z_{\alpha,0}).$$

Here the first identity uses the evenness of $F_{1/s}$. If $r \in [1, p) \setminus I$, the two samples are independent and hence give $|f_y(\varepsilon_r z_{\alpha,r})|^2$. If $r \in I \setminus \{0\}$, the same sample occurs in both copies and $|\psi_{\alpha,r}(X)|^2 = 1$; thus both its character and its target phase cancel. At $r = 0$, the resulting factor is $\|G_y(z_{\alpha,0})\|^2$ when $0 \notin I$, and $\mathbb{E}_{X \sim D_{\mathcal{L}^*,s}}[\|2\pi X\|^2]$ when $0 \in I$. Consequently,

$$\mathbb{E}_{\delta,X}[(\mathcal{K}_i^y(\theta))^* \mathcal{K}_{i'}^y(\theta) \mid U] = \begin{cases} T_{m-1,I,y}(\theta), & 0 \notin I, \\ \mathbb{E}_{X \sim D_{\mathcal{L}^*,s}}[\|2\pi X\|^2] V_{m-1,I,y}(\theta), & 0 \in I. \end{cases} \quad (97)$$

Splitting the expectation according to $\|X\| \leq Cs\sqrt{n}$ and its complement, and applying [Lemma 3.2](#), gives

$$\mathbb{E}_{X \sim D_{\mathcal{L}^*,s}}[\|2\pi X\|^2] \leq O(ns^2) \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{o(n)}.$$

Therefore, [Lemma 5.16](#) bounds the absolute value of [Eq. \(97\)](#) by the right-hand side of [Eq. \(96\)](#).

Applying the same character orthogonality to two independent copies gives

$$\mathbb{E}_\delta[\|A_{y,\delta}(\theta)\|^2 \mid U] = T_{m-1,y}(\theta) \leq \sum_{\kappa \in R_{m-1}} T_{m-1,y}(\kappa) \leq \left(\frac{s^2 \lambda}{q}\right)^2 2^{o(n)}, \quad (98)$$

where $R_{m-1} = H_m$ and we used [Eq. \(93\)](#). Expanding the two subtractions in the expectation in the lemma shows that it is the difference between [Eqs. \(97\) and \(98\)](#). Since $I \neq \emptyset$, the triangle inequality proves [Eq. \(96\)](#), after absorbing a factor of two into $2^{o(n)}$. $\square$

Proof of Lemma 5.6. This is the same counting argument as in the proof of [Lemma 4.6](#). Work on the event in [Lemma 5.12](#), on which [Lemma 5.17](#) holds. For a fixed nonempty $I \subseteq [0, p)$, at most $N^{2p-|I|}$ ordered pairs (i, i') have overlap set I . Expanding the squared norm of the estimator and applying [Eq. \(96\)](#) give

$$\begin{aligned} & \mathbb{E}_{\delta,X} \left[\left\| \widehat{A}_{y,\delta}(\theta) - A_{y,\delta}(\theta) \right\|^2 \mid U \right] \\ & \leq \frac{1}{N^{2p}} \sum_{\emptyset \neq I \subseteq [0,p)} N^{2p-|I|} \left(\frac{s^2 \lambda}{q}\right)^2 Q^{|I|-1} 2^{o(n)} \\ & = \left(\frac{s^2 \lambda}{q}\right)^2 Q^{-1} 2^{o(n)} \left[\left(1 + \frac{Q}{N}\right)^p - 1 \right] \leq \left(\frac{s^2 \lambda}{q}\right)^2 Q^{-1} 2^{o(n)}, \end{aligned}$$

where the last inequality follows from $N = Qp^4$. The event is uniform in y and θ , proving the lemma. $\square$

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