

Two Standard Deviations Are Necessary for the Kadison-Singer Problem

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Abstract

Let C_0 be the least constant such that, for every finite family of vectors $u_1, \dots, u_n \in \mathbb{C}^d$ and independent finitely supported real random variables ξ_i , there are values $\varepsilon_i \in \text{supp}(\xi_i)$ satisfying $\|\sum_{i=1}^n (\varepsilon_i - \mathbb{E}[\xi_i]) u_i u_i^*\| \leq C_0 \|\sum_{i=1}^n \text{Var}[\xi_i] (u_i u_i^*)^2\|^{1/2}$, where $\|\cdot\|$ denotes the operator norm. This rank-one matrix discrepancy formulation generalizes the signing formulation of the Kadison–Singer problem [KS59], resolved by Marcus, Spielman, and Srivastava [MSS15b].

Let C_1 be the least constant such that, for every $\epsilon > 0$ and every finite family in $\mathbb{C}^d$, in every dimension d , satisfying $\sum_{i=1}^n u_i u_i^* = I$ and $\max_{i \in [n]} \|u_i\|^2 \leq \epsilon$, there are signs $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that $\|\sum_{i=1}^n \varepsilon_i u_i u_i^*\| \leq C_1 \sqrt{\epsilon}$. Taking the ξ_i to be independent symmetric signs gives $C_1 \leq C_0$, since $\|\sum_{i=1}^n (u_i u_i^*)^2\| \leq \epsilon$ under these hypotheses. The general formulation allows arbitrary finite real supports and does not require $\sum_{i=1}^n u_i u_i^* = I$. Kyng, Luh, and Song [KLS20] proved $C_0 \leq 4$. We prove $2 \leq C_1 \leq C_0 \leq 2.176$. This improves the bounds $\sqrt{2} \leq C_0 \leq 3$ of Xie, Xu, and Zhu [XXZ21].

We conjecture that the optimal constant is $C_0 = 2$.

1 Introduction

For every positive integer m , write $[m] := \{1, \dots, m\}$. Let $A_i := u_i u_i^*$ for vectors $u_i \in \mathbb{C}^d$, and let $\xi_1, \dots, \xi_n$ be independent real random variables with finite support. The rank-one matrix discrepancy problem asks how closely the mean $\sum_{i=1}^n \mathbb{E}[\xi_i] A_i$ can be approximated by choosing one value from each support. The discrepancy and its variance parameter are

$$\text{Disc}(A; \xi) := \min_{\varepsilon_i \in \text{supp}(\xi_i)} \left\| \sum_{i=1}^n (\varepsilon_i - \mathbb{E}[\xi_i]) A_i \right\|, \quad (1)$$

$$\sigma(A; \xi)^2 := \left\| \sum_{i=1}^n \text{Var}[\xi_i] A_i^2 \right\|. \quad (2)$$

Here and throughout, $\|\cdot\|$ denotes the operator norm. Let C_0 be the least constant for which $\text{Disc}(A; \xi) \leq C_0 \sigma(A; \xi)$ holds for every such family in every dimension. Definition 2.1 records these conventions formally.

Kadison and Singer [KS59] posed their pure-state extension problem in 1959. Weaver [Wea04] reformulated it as a rank-one discrepancy problem. Marcus, Spielman, and Srivastava [MSS15a, MSS15b] resolved it using interlacing families and multivariate barrier estimates. For independent

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symmetric signs with $\sum_{i=1}^n A_i = I$ and $\epsilon := \max_{i \in [n]} \|A_i\| \leq 1$, their bipartition bound [MSS15b, Corollary 1.5] gives $\text{Disc}(A; \xi) \leq 2\sqrt{2\epsilon} + 2\epsilon$. The stability properties of determinant polynomials established by Borcea and Brändén [BB08] underpin this method.

Akemann and Weaver [AW14] proved an approximate Lyapunov theorem for rank-one matrices. Under $\sum_{i=1}^n A_i \preceq I$ and $\|A_i\| \leq \epsilon$, they approximated each sum $\sum_{i=1}^n t_i A_i$, with $t_i \in [0, 1]$, by a subset sum with operator norm error $O(\epsilon^{1/8})$. Kyng, Luh, and Song [KLS20, Corollary 1.8] reduced this error to $2\sqrt{\epsilon}$.

Anari and Oveis Gharan [AO15] developed second-order barrier estimates for strongly Rayleigh measures. Kyng, Luh, and Song [KLS20] adapted these estimates and proved $C_0 \leq 4$. Their theorem permits arbitrary finite real supports and arbitrary rank-one families. Under the MSS hypotheses, $\sigma(A; \xi)^2 \leq \epsilon$, and the variance can be much smaller than ϵ . The bound $4\sigma(A; \xi)$ therefore extends the scope of MSS and can improve its discrepancy estimate. Its specialization $4\sqrt{\epsilon}$ has a larger leading coefficient than the MSS bound as $\epsilon \rightarrow 0$ [KLS20, Corollary 1.6].

Xie, Xu, and Zhu [XXZ21] reduced the upper bound to 3 and proved $C_0 \geq \sqrt{2}$. Their unit-norm tight frames with $n = 2d - 1$ have discrepancy ratio $\sqrt{2 - 1/d}$ for independent symmetric random signs. This ratio tends to $\sqrt{2}$ as the dimension grows.

Closing the gap between the upper and lower bounds for the Kadison–Singer constant is mentioned as an interesting open problem in [Son19].

Three recent independent and concurrent works [EJ26, Kat26, SY26] give polynomial-time algorithms for the Kadison–Singer problem, achieving constants 13, 35, and 3.34, respectively.

1.1 Our results

We state our result as follows.

Theorem 1.1 (Informal versions of Theorems 4.1 and 5.1). *Let C_0 be the least $C \geq 0$ for which $\text{Disc}(A; \xi) \leq C\sigma(A; \xi)$ holds for every finite family $A_i := u_i u_i^*$, with $u_i \in \mathbb{C}^d$, in every dimension d , and every choice of independent finitely supported real random variables ξ_i . Then*

$$2 \leq C_0 \leq 2.176.$$

This improves the upper bound of Xie, Xu, and Zhu [XXZ21] from 3 to 2.176 and their lower bound from $\sqrt{2}$ to 2. We conjecture that the optimal constant is $C_0 = 2$. Recent work [SZ26] also establishes a lower bound of 2 for the matrix Spencer problem. We expect 2 to be the optimal constant in that setting as well.

Next we consider independent symmetric signs and isotropic families, with discrepancy measured on the $\sqrt{\epsilon}$ scale.

Theorem 1.2 (Informal versions of Theorems 4.1 and 6.1). *Let C_1 be the least $C \geq 0$ for which $\text{Disc}(A; \xi) \leq C\sqrt{\epsilon}$ holds for every finite family $A_i := u_i u_i^*$, with $u_i \in \mathbb{C}^d$, in every dimension d , satisfying $\sum_{i=1}^n A_i = I_d$, where $\epsilon := \max_{i \in [n]} \|u_i\|^2$ and the ξ_i are independent symmetric signs. Then*

$$2 \leq C_1 \leq C_0 \leq 2.176.$$

Finally, we express the signing bounds in Weaver’s two-partition formulation.

Theorem 1.3 (Informal versions of Theorem 4.1 and Corollary 6.3). *Let C_W be the least $C \geq 0$ such that every finite family $A_i := u_i u_i^*$, with $u_i \in \mathbb{C}^d$, in every dimension d , satisfying $\sum_{i=1}^n A_i = I_d$*

admits a partition $[n] = S_1 \dot{\cup} S_2$ with

$$\max_{j \in [2]} \left\| \sum_{\substack{i=1 \\ i \in S_j}}^n A_i \right\| \leq \frac{1}{2} + C\sqrt{\epsilon}, \quad \epsilon := \max_{i \in [n]} \|u_i\|^2.$$

Then

$$1 \leq C_W = \frac{C_1}{2} \leq \frac{C_0}{2} \leq 1.088.$$

We conjecture that $C_W = 1$.

Table 1: Discrepancy bounds for independent symmetric signs with $\sum_{i=1}^n A_i = I$ and $\epsilon := \max_{i \in [n]} \|A_i\| \leq 1$. Panel (b) gives a worst-case asymptotic lower bound as $\epsilon \rightarrow 0$ in the sense of Theorem 6.1. Numerical coefficients are rounded outward to three decimal places.

(a) Upper bounds			
Year	Authors	Reference	Discrepancy bound
2015	Marcus, Spielman, and Srivastava	[MSS15b]	$2.829\sqrt{\epsilon} + 2\epsilon$
2020	Kyng, Luh, and Song	[KLS20]	$4.000\sqrt{\epsilon}$
2021	Xie, Xu, and Zhu	[XXZ21]	$3.000\sqrt{\epsilon}$
2026	This paper	Theorem 4.1	$2.176\sqrt{\epsilon}$

(b) Lower bound			
Year	Authors	Reference	Worst-case lower bound
2026	This paper	Theorem 6.1	$(2 - o(1))\sqrt{\epsilon}$

Table 2: Bounds for the optimal universal constant C_0 in $\text{Disc}(A; \xi) \leq C_0 \sigma$, where $\sigma := \|\sum_{i=1}^n \text{Var}[\xi_i] A_i^2\|^{1/2}$. Upper bounds are rounded upward and lower bounds downward to three decimal places.

(a) Upper bounds			
Year	Authors	Reference	Constant C_0
2020	Kyng, Luh, and Song	[KLS20]	4.000
2021	Xie, Xu, and Zhu	[XXZ21]	3.000
2026	This paper	Theorem 4.1	2.176

(b) Lower bounds			
Year	Authors	Reference	Constant C_0
2021	Xie, Xu, and Zhu	[XXZ21]	1.414
2026	This paper	Theorem 5.1	2.000

Organization. Section 2 introduces the polynomial preliminaries. Section 3 gives the technical overview. Section 4 proves the upper bound using moment inequalities and rational shift schedules. Section 5 proves the variance-normalized lower bound. Section 6 proves the lower bound in the $\sqrt{\epsilon}$ normalization. Appendix A records the longer schedule coefficients, explicit Gram witnesses, and their verification.

2 Preliminaries

Section 2.1 introduces discrepancy, real stable polynomials, and basic barrier estimates. Section 2.2 proves the local barrier update. Section 2.3 introduces the homogeneous polynomial and proves the cone comparison.

2.1 Discrepancy and real stable polynomials

We first fix the discrepancy and variance normalization used throughout the paper.

Definition 2.1 (Rank-one discrepancy). *Let $A_i := u_i u_i^*$ for $u_i \in \mathbb{C}^d$, $i \in [n]$, and let ξ_i be independent real random variables with finite support. With operator norms, define $\sigma(A; \xi) := \|\sum_{i=1}^n \text{Var}[\xi_i] A_i^2\|^{1/2}$ and*

$$\text{Disc}(A; \xi) := \min_{\varepsilon_i \in \text{supp}(\xi_i)} \left\| \sum_{i=1}^n (\varepsilon_i - \mathbb{E}[\xi_i]) A_i \right\|.$$

We abbreviate $\sigma(A; \xi)$ to σ when the family is fixed. The universal constant C_0 is the least $C \geq 0$ such that $\text{Disc}(A; \xi) \leq C\sigma(A; \xi)$ for every such family.

We next introduce the polynomial domain and logarithmic derivatives used in the barrier argument.

Definition 2.2. *A nonzero polynomial $p \in \mathbb{R}[z_1, \dots, z_m]$ is real stable if $p(z) \neq 0$ whenever $\text{Im } z_i > 0$ for every i . Define*

$$\text{Ab}_p := \{a \in \mathbb{R}^m : p(a + t) > 0 \text{ for every } t \in \mathbb{R}_{\geq 0}^m\}.$$

For $a \in \text{Ab}_p$, its i th barrier is $\Phi_p^i(a) := \partial_i p(a)/p(a)$.

Real specialization preserves stability unless the resulting polynomial is zero; a positive value at the reference point excludes that exception.

The bivariate representation used below is an external input. In the normalization of [BCMS19, Corollary 3.4], if $p(x, y)$ is real stable and $(x_0, y_0) \in \text{Ab}_p$, there are real symmetric positive semidefinite matrices A, B such that

$$\frac{p(x_0 + s, y_0 + t)}{p(x_0, y_0)} = \det(I + sA + tB). \quad (3)$$

Constant polynomials use the empty determinant. Indeed, the cited representation gives a positive definite matrix M at the reference point, and congruence by $M^{-1/2}$ gives Eq. (3).

These estimates control barriers under coordinate shifts and give the stability-preserving operators used below.

Lemma 2.3. *Let p be real stable and $a \in \text{Ab}_p$. Then, for $t \geq 0$ coordinatewise,*

$$0 \leq \Phi_p^i(a + t) \leq \Phi_p^i(a), \quad (4)$$

$$0 \leq \frac{\partial_i^2 p(a + t)}{p(a + t)} \leq \Phi_p^i(a + t)^2. \quad (5)$$

For every real c , $(1 + c\partial_i)p$ is real stable.

Proof. Fix the other coordinates at a real point above the roots. The resulting univariate polynomial is nonzero and has real roots strictly to the left of the evaluation point. If the positive distances to its roots, counted with multiplicity, are $r_1, \dots, r_\ell$, logarithmic differentiation gives

$$\Phi_p^i = \sum_{h=1}^{\ell} r_h^{-1}, \quad (6)$$

$$\frac{\partial_i^2 p}{p} = \left(\sum_{h=1}^{\ell} r_h^{-1} \right)^2 - \sum_{h=1}^{\ell} r_h^{-2}. \quad (7)$$

The empty sums handle degree zero. Eq. (6) is nonnegative. The expression in Eq. (7) is the sum of the nonnegative terms $r_h^{-1} r_k^{-1}$ with $h \neq k$, and is at most the square in Eq. (6).

For distinct coordinates i, j , restrict to those coordinates and use Eq. (3). Differentiating the logarithm of the determinant at the reference point yields $\partial_j \Phi_p^i = -\text{tr}[AB] \leq 0$, because $A, B \succeq 0$. For $j = i$, differentiation of Eq. (6) gives $\partial_i \Phi_p^i = -\sum_{h=1}^{\ell} r_h^{-2} \leq 0$. Integrate these inequalities along coordinate segments inside Ab_p to obtain Eq. (4).

To prove the last assertion, fix every other variable in the open upper half-plane. The univariate polynomial in z_i has all its roots in the closed lower half-plane. If it is nonconstant, its logarithmic derivative has strictly negative imaginary part at every z_i in the upper half-plane, since each summand $1/(z_i - r_h)$ does. Consequently $1 + c\partial_i p/p$ cannot vanish for nonzero real c . The constant case and $c = 0$ are immediate. Thus $(1 + c\partial_i)p$ has no zero when all coordinates have positive imaginary parts. $\square$

The upper-bound argument will control the largest zero of the following polynomial.

Definition 2.4 (Discrepancy polynomial). *For a finite family of positive semidefinite rank-one matrices $B_1, \dots, B_n$, define $V := \sum_{i=1}^n B_i^2$ and*

$$P_B(x) := \left[\prod_{i=1}^n \left(1 - \frac{1}{2} \partial_{z_i}^2 \right) \det \left(xI + \sum_{i=1}^n z_i B_i \right)^2 \right]_{z=0}. \quad (8)$$

Let $\lambda_{\max}(P_B)$ denote its largest real zero. A constant $c \geq 0$ is admissible if, for every such family,

$$\lambda_{\max}(P_B) \leq c \|V\|^{1/2}. \quad (9)$$

2.2 The local barrier update

The next lemma converts a directional eigenvalue bound into a sufficient shift for a second-order barrier update.

Lemma 2.5. *Let p be real stable. Suppose $a \in \text{Ab}_p$ and $\deg_i p \leq 2$. Put $u := \Phi_p^i(a)$. For each $j \neq i$, let A and B be the matrices representing the (i, j) restriction in Eq. (3), with A corresponding to coordinate i . These matrices may depend on j . Assume*

$$A^2 \succeq \ell A$$

for some $\ell \geq 0$ independent of j . If $\delta > 0$ satisfies

$$u \leq 2\delta, \quad \delta \geq u - \ell, \quad (10)$$

then, for $q := (1 - \frac{1}{2}\partial_i^2)p$,

$$a + \delta e_i \in \text{Ab}_q, \quad \Phi_q^j(a + \delta e_i) \leq \Phi_p^j(a) \quad (j \neq i).$$

Here $\ell = 0$ is always allowed; if the direction has degree two, ℓ may be the smaller of its two positive directional eigenvalues.

Proof. The factorization $1 - \frac{1}{2}\partial_i^2 = (1 - \partial_i/\sqrt{2})(1 + \partial_i/\sqrt{2})$ preserves real stability. Fix $b := a + t$, $t \geq 0$. Write $p(b + se_i)/p(b) = (1 + \alpha s)(1 + \beta s)$ with $\alpha, \beta \geq 0$, allowing a zero factor. Let $U := \alpha + \beta \leq u \leq 2\delta$. Then

$$\frac{q(b + \delta e_i)}{p(b)} = 1 + \delta U + (\delta^2 - 1)\alpha\beta.$$

For $\delta \geq 1$ this is positive. For $0 < \delta < 1$, it is at least $1 + \delta U - U^2/4 \geq 1 + \delta U/2 > 0$. This proves positivity on the entire translated positive orthant.

In Eq. (3), put $v := \text{tr}[B]$, $w := \text{tr}[AB]$ and $r := \text{tr}[A^2B]$; then $u = \text{tr}[A]$. If $R(s, t) := q(a + (s + \delta)e_i + te_j)/p(a)$, exact differentiation gives

$$R_t(0, 0) - vR(0, 0) = -\delta w + (1 - \delta^2)(uw - r). \quad (11)$$

Indeed, for $F := \det(I + sA + tB)$, at the origin $F_{st} = uv - w$, $F_{ss} = u^2 - \text{tr}[A^2]$, and $F_{sst} = vF_{ss} - 2uw + 2r$; also $R = F + \delta F_s + (\delta^2 - 1)F_{ss}/2$. Since $uA - A^2 \succeq 0$, we have $0 \leq uw - r$. For $\delta \geq 1$, Eq. (11) is nonpositive. Otherwise $r \geq \ell w$, and its right side is at most $[-\delta + (1 - \delta^2)(u - \ell)]w \leq 0$ by Eq. (10). Dividing by $R(0, 0) > 0$ proves the claim. $\square$

2.3 The homogeneous polynomial and cone comparison

For a collection $(D_j)_{j \in J}$ of PSD matrices of rank at most one, let

$$H_J(X, t) := \prod_{j \in J} (1 - \frac{t^2}{2} \mathcal{D}_{D_j}^2) \det(X)^2, \quad \mathcal{D}_D f(X) := \frac{d}{ds} f(X + sD)|_{s=0}. \quad (12)$$

This is a real homogeneous polynomial of degree $2d$ on the real vector space of Hermitian matrices X together with a real scalar t . It is hyperbolic in direction $e := (I, 0)$: for every fixed Hermitian X and real t , stability of $\det(X + xI + \sum_{j \in J} z_j D_j)^2$ and the factored differential operators imply that $x \mapsto H_J(X + xI, t)$ is real-rooted. Its leading coefficient is one.

Let Γ_J be its open hyperbolicity cone containing $(I, 0)$. We use these standard facts from Gårding's theory [HHBL10, Theorem 2.9, Corollary 2.13, Theorem 2.15]: Γ_J is convex; every point of Γ_J is a hyperbolicity direction; and ordered hyperbolic eigenvalues are monotone under addition of elements of $\bar{\Gamma}_J$ (by continuity at the boundary).

Processed polynomials and cone membership. The schedules below use both real stability and membership in Γ_J . To relate them, fix a Hermitian matrix X_0 and PSD rank-one directions $E_1, \dots, E_m$, and define

$$q_J(x, z) := H_J(X_0 + xI + \sum_{i=1}^m z_i E_i, 1).$$

This polynomial is real stable. Indeed, introduce auxiliary variables y_j for $j \in J$ and start with

$$\det(X_0 + xI + \sum_{i=1}^m z_i E_i + \sum_{j \in J} y_j D_j)^2.$$

When all variables have positive imaginary parts, the matrix inside has positive definite imaginary part because of xI , so its determinant is nonzero. Apply $\prod_{j \in J} (1 - \partial_{y_j}^2/2)$, factoring each operator into two real first-order stability-preserving operators, and then specialize all y_j to zero. The result is q_J . It is not the zero polynomial: its leading coefficient in x is one and its degree in x is $2d$. Its degree in each z_i is at most two.

At a real point $a = (x, z)$, write $M := X_0 + xI + \sum_{i=1}^m z_i E_i$. Then

$$a \in \text{Ab}_{q_J} \iff (M, 1) \in \Gamma_J.$$

For the forward implication, increase only the spectral coordinate: $H_J(M + tI, 1) > 0$ for every $t \geq 0$. Hyperbolicity makes all roots of $t \mapsto H_J(M + tI, 1)$ real, so they are strictly negative. This is precisely the root characterization of the cone containing $(I, 0)$. For the converse, every PSD direction $(K, 0)$ lies in $\bar{\Gamma}_J$: positive definite K is connected to I through positive definite matrices at scalar coordinate zero, where $H_J(K, 0) = \det(K)^2 > 0$, and singular K follows by a limit. Thus adding any nonnegative combination of $(I, 0)$ and $(E_i, 0)$ to $(M, 1)$ stays in Γ_J , by convexity of the cone. The polynomial is positive there, proving positivity on the entire translated nonnegative orthant. This equivalence supplies initial above-root membership from an initial cone estimate and transfers every maintained above-root condition back to the cone after a processed operator is added.

The next comparison lets us enlarge the matrix argument when estimating directional eigenvalues.

Lemma 2.6. *For any hyperbolic polynomial H with cone Γ , suppose $a \in \bar{\Gamma}$, $b, c \in \Gamma$ and $c - b \in \bar{\Gamma}$. Then its ordered nonnegative eigenvalues satisfy*

$$\lambda_k^b(a) \geq \lambda_k^c(a) \quad \text{for every } k.$$

Here $H(b + sa)/H(b) = \prod_{k=1}^{\deg H} (1 + s\lambda_k^b(a))$.

Proof. Fix an interior reference direction e and order its eigenvalues decreasingly. For $t \geq 0$, cone monotonicity gives $\lambda_k^e(a - tb) \geq \lambda_k^e(a - tc)$. Each of these functions is strictly decreasing through zero; its zero is the corresponding ordered eigenvalue $\lambda_k^b(a)$ or $\lambda_k^c(a)$, respectively. This follows either from the change-of-direction assertion in [HHBL10, Theorem 2.9], or directly by counting the zeros of $H(a - tb)$ and $H(a - tc)$. The zero for the larger function occurs no earlier. Boundary values of a follow by continuity. $\square$

3 Technical overview

Section 3.1 derives the isotropic signing upper bound from the variance bound and explains its consequences for partitions and subset approximation. Section 3.2 shows how covariance normalization converts Gaussian families into isotropic lower-bound examples. Section 3.3 explains the nested inductions that turn an admissible upper bound into a sharper one. Section 3.4 develops the directional barrier estimates used in these improvements, and Section 3.5 combines Gaussian trace moments and concentration to construct examples with large discrepancy for every signing.

3.1 Upper bounds for isotropic signings

Step 1: Normalize the variance. Let $A_i := u_i u_i^*$, where $\sum_{i=1}^n A_i = I$ and $\max_i \|u_i\|^2 \leq \epsilon$. Theorem 4.1 gives a signing whose operator norm is at most $c_* \sqrt{\epsilon}$, where $c_* := 108751/50000$. The

passage to this normalization uses the rank-one identity $A_i^2 = \|u_i\|^2 A_i$. For independent symmetric signs, the variance parameter therefore satisfies

$$\sigma^2 = \left\| \sum_{i=1}^n A_i^2 \right\| = \left\| \sum_{i=1}^n \|u_i\|^2 A_i \right\| \leq \epsilon. \quad (13)$$

The first step is the variance definition for symmetric signs, the second uses rank one, and the last follows from isotropy and the bound on each vector norm. Applying the general discrepancy theorem gives the claimed square-root bound.

Step 2: Encode both spectral extremes. The spectral argument controls both ends of the spectrum of a signed sum. For $s \in \{-1, 1\}^n$, define $S_s := \sum_{i=1}^n s_i A_i$ and $p_s(x) := \det(xI - S_s) \det(xI + S_s)$. The largest zero of p_s is exactly $\|S_s\|$. The interlacing method of Marcus, Spielman, and Srivastava [MSS15a, MSS15b], in the discrepancy formulation of Kyng, Luh, and Song [KLS20], selects a signing whose largest zero is at most that of the average polynomial $\mathbb{E}_s[p_s]$. Real stability ensures that the conditional averages used in this selection are real-rooted. Thus a bound for one average polynomial yields a signing that controls the positive and negative eigenvalues simultaneously.

Step 3: Reduce the barrier shifts. Our improvement comes from the barrier estimate for this average polynomial. A barrier is a logarithmic derivative at a point above the roots; it measures how far the point must move when a differential operator is applied. In the second-order framework of Anari and Oveis Gharan [AO15] and Kyng, Luh, and Song [KLS20], each coordinate has degree at most two. Lemma 2.5 uses the smaller of its two directional eigenvalues to reduce the required shift. A determinantal representation in three variables [LPR05] makes this eigenvalue accessible through a rank-two compression. Lemma 4.2 then gives a positive lower bound for it, uniformly in the ambient dimension.

Step 4: Sharpen the root bound. We improve the root bound by reducing both the initial barrier and the shifts needed to maintain it. For each direction, the determinantal representation expresses the barrier through resolvent averages, which weight spectral values more strongly as they approach the evaluation point. Relations among the directional moments through order twelve restrict how much spectral weight can concentrate near this point. Positive square and Gram-matrix remainders strengthen these relations. Combining them with a polynomial upper bound for the resolvent gives a smaller initial barrier (Lemma 4.37).

The compression estimate improves the shifts themselves. We retain positive Schur-complement corrections and use information from the directional Hessian to obtain a stronger lower bound for the smaller directional eigenvalue. If the current barrier is u and this eigenvalue is ℓ , a shift δ suffices when $u \leq 2\delta$ and $\delta \geq u - \ell$ (Lemma 2.5). Thus a smaller barrier and a larger lower bound for ℓ both help reduce the required shift. A polynomial schedule specifies the shifts throughout the argument. Its initial point and total shift give the resulting root bound, while exact rational certificates verify that the shift conditions hold at every stage.

We then feed each improved root bound back into the estimates. A smaller bound confines the directional spectrum to a shorter interval and sharpens the next initial barrier. Comparing the barrier before and after a complete round of differential operators gives additional inequalities for rational functions of the same spectral variable. Combining these resolvent inequalities with the moment relations strengthens the iteration further. The resulting schedules yield the final coefficient c_* (Section 3.4). Since $\sigma \leq \sqrt{\epsilon}$, the variance bound $c_*\sigma$ gives the signing bound $c_*\sqrt{\epsilon}$, with upper coefficient 2.176 after outward rounding.

Step 5: Recover partition and subset bounds. The same argument yields partition and subset-approximation bounds. For the set $S := \{i : s_i = 1\}$, isotropy gives $\sum_{i \in S}^n A_i - I/2 = S_s/2$.

Both parts of the resulting partition have operator norm at most $1/2 + (c_*/2)\sqrt{\epsilon}$. For arbitrary target weights $t_i \in [0, 1]$, Bernoulli variables of means t_i have variance at most $1/4$. Corollary 4.45 therefore approximates $\sum_{i=1}^n t_i A_i$ by a subset sum with error at most $(c_*/2)\sqrt{\epsilon}$. This conclusion also holds under $\sum_{i=1}^n A_i \preceq I$ and gives the outward-rounded coefficient 1.088.

3.2 Lower bounds by covariance normalization

Step 1: Construct a Gaussian family. Theorem 6.1 constructs, for every $0 < c < 2$ and every sufficiently small prescribed $\epsilon > 0$, an isotropic real family with maximum squared vector norm exactly ϵ such that every signing has norm greater than $c\sqrt{\epsilon}$. The construction begins with the Gaussian family used for the variance lower bound. Its covariance is close to a scalar matrix. Normalizing by this covariance makes the family exactly isotropic while preserving the discrepancy up to an explicit factor tending to one.

Let $g_1, \dots, g_n \in \mathbb{R}^d$ be independent standard Gaussian vectors, let $n := \alpha d$, and set $a_i := g_i/\sqrt{d}$. Define $H := \sum_{i=1}^n a_i a_i^\top$ and $D(a) := \min_{s \in \{-1, 1\}^n} \|\sum_{i=1}^n s_i a_i a_i^\top\|$. For the Catalan number $\text{Cat}_k := \binom{2k}{k}/(k+1)$, let $m_k := (\text{Cat}_k/2)^{1/(2k)}$; then $m_k \rightarrow 2$. Lemmas 5.3 and 5.5 give realizations with $D(a) \geq m_k \sqrt{\alpha}$, covariance between $\alpha(1-\delta)I$ and $\alpha(1+\delta)I$, and $\max_i \|a_i\|^2 \leq 1+\delta$. Here $k, \delta > 0$, and a sufficiently large integer α are fixed before d tends to infinity. The simultaneous estimate over all signings is explained in Section 3.5.

Step 2: Normalize the covariance. Set $v_i := H^{-1/2} a_i$ and define $D(v)$ by the same minimum over signings. Lemma 6.2 gives

$$\sum_{i=1}^n v_i v_i^\top = I, \quad (14)$$

$$\max_i \|v_i\|^2 \leq \frac{1+\delta}{\alpha(1-\delta)}, \quad (15)$$

$$D(v) \geq \frac{m_k}{(1+\delta)\sqrt{\alpha}}. \quad (16)$$

The covariance identity follows directly by congruence. The norm bound uses the smallest eigenvalue of H . For the discrepancy bound, each original signed sum equals $H^{1/2}(\sum_{i=1}^n s_i v_i v_i^\top)H^{1/2}$, so its norm is at most $\|H\|$ times the normalized signed norm. This comparison preserves the lower bound for every signing.

Step 3: Match the prescribed norm cap. To meet the prescribed cap, put $\rho := (1+\delta)/(1-\delta)$ and choose $\alpha := \lceil \rho/\epsilon \rceil$. Eq. (15) gives $\max_i \|v_i\|^2 \leq \epsilon$. The ceiling inequality $\alpha\epsilon \leq \rho + \epsilon$ and Eq. (16) give

$$\frac{D(v)}{\sqrt{\epsilon}} \geq \frac{m_k}{(1+\delta)\sqrt{\alpha\epsilon}} \geq \frac{m_k}{(1+\delta)\sqrt{\rho + \epsilon}}. \quad (17)$$

The first step applies the normalized discrepancy estimate, and the last uses the ceiling inequality. For fixed k and δ , the final expression tends to $m_k \sqrt{1-\delta}/(1+\delta)^{3/2}$ as $\epsilon \rightarrow 0$. Choosing k large and then δ small makes this limit exceed any fixed $c < 2$. All sufficiently small ϵ retain a strict margin.

Step 4: Make the norm cap exact. The exact equality in the norm cap is obtained by adjoining an orthogonal coordinate. Split its unit mass into finitely many positive pieces of size at most ϵ , including one piece equal to ϵ , and use the corresponding collinear vectors on that coordinate. The enlarged family remains isotropic. Every signed sum contains the previous signed sum as a diagonal block, so its norm retains the lower bound. This completes the mechanism behind Theorem 6.1.

Step 5: Transfer the lower bound to subsets. Finally, taking target weights $t_i = 1/2$ divides every signed sum by two. Corollary 6.3 therefore gives a lower asymptotic coefficient of 1 for Lyapunov subset approximation, and a partition lower bound of $1/2 + (c/2)\sqrt{\epsilon}$ for every $c < 2$.

3.3 Double induction for the upper bound

Step 1: Establish the starting bound. The upper-bound proof has two nested inductions. The outer induction improves a universal admissible constant. The inner induction preserves the polynomial barrier conditions while processing auxiliary copies of the matrix directions. Section 4.2 supplies the starting admissible bound, and the fourth-moment argument in Section 4.3 gives the first instance of this refinement.

Step 2: Start at a smaller scale. Suppose $C > \sqrt{2}$ is already admissible. By Lemma 4.6, it suffices to consider an arbitrary family normalized by $\sum_{i=1}^n B_i^2 = I$. For $0 < s_0 < 1$, scaling gives

$$\lambda_{\max}(P_{s_0 B}) = s_0 \lambda_{\max}(P_B) \leq C s_0. \quad (18)$$

Thus a scalar point $m_0 I$ with $m_0 > C s_0$ starts above the roots of the scaled polynomial. Lemma 4.8 also uses C to bound the initial directional barriers. Since $s_0 < 1$, the starting point can satisfy $m_0 < C$; the improvement depends on the additional movement needed to reach full scale.

Step 3: Apply the inner induction. The inner induction establishes this movement bound for a fixed family. In Lemma 4.9, divide the remaining variance into N rounds and process the auxiliary copies one at a time. Initially, Lemma 4.8 supplies above-root membership and the bounds on unused barriers. At each update, the compression estimate of Lemma 4.2 bounds the smaller directional eigenvalue ℓ . The schedule chooses a shift δ satisfying $u \leq 2\delta$ and $u - \ell \leq \delta$, where u is the current barrier. Lemma 2.5 then preserves above-root membership and all unused barrier bounds. These properties form the induction invariant.

Step 4: Obtain an improved bound. After all copies have been processed, their operators combine with the initial operators to give the original polynomial P_B . Its largest root is bounded by the finite schedule endpoint. Letting $N \rightarrow \infty$ gives

$$C' := m(1) = m_0 + \int_{s_0}^1 2sk(s) \, ds, \quad (19)$$

where k is the continuous shift rate. The strict schedule margins justify this passage through uniform Riemann-sum convergence, as in Lemmas 4.5 and 4.9. Because the family was arbitrary, C' is admissible. Bounding the integral in Eq. (19) by less than $C - m_0$ therefore proves a strictly smaller universal bound.

Step 5: Apply the outer induction. The outer induction reuses this improved constant to restrict the directional spectrum and sharpen the next initial barrier. For example, Section 4.3 constructs an admissible endpoint below $c_4 + (C - c_4)/4$ from each admissible input C in its specified interval, where $c_4 := 4758489/2000000$. Starting from its already admissible input $C_{\text{in},4}$, induction gives admissibility of

$$\widehat{C}_j := c_4 + 4^{-j}(C_{\text{in},4} - c_4) \quad (j \geq 0). \quad (20)$$

Taking $j \rightarrow \infty$ proves that c_4 is admissible. The limit in the number of auxiliary rounds establishes each improvement; the limit in j establishes the endpoint of this outer iteration. The final resolvent refinements use a finite sequence of certified improvements through Lemma 4.43 to reach the bound c_* .

3.4 Variance upper bounds from directional barriers

Step 1: Normalize the variables and encode discrepancy. For arbitrary finite real supports, let $A_i := u_i u_i^*$, $\mu_i := \mathbb{E}[\xi_i]$, $\tau_i^2 := \text{Var}[\xi_i]$, and $\sigma^2 := \|\sum_{i=1}^n \tau_i^2 A_i^2\|$. When $\sigma > 0$, omit deterministic variables and set $B_i := \tau_i A_i / \sigma$ and $\eta_i := (\xi_i - \mu_i) / \tau_i$. Then $V := \sum_{i=1}^n B_i^2 \preceq I$. Rank one makes each determinant affine in each variable, giving the discrepancy polynomial of Definition 2.4:

$$P_B(x) := \mathbb{E}[\det(x^2 I - (\sum_{i=1}^n \eta_i B_i)^2)] = [\prod_i (1 - \frac{1}{2} \partial_{z_i}^2) \det(xI + \sum_{i=1}^n z_i B_i)^2]_{z=0}. \quad (21)$$

Only the means and variances enter this identity. The interlacing selection of Kyng, Luh, and Song [KLS20] transfers its largest-root bound to choices in the original supports, as proved in Section 4. Rescaling then restores the factor σ .

Step 2: Control the local shifts. The main task is to keep a reference point above the roots while applying the second-order operators. If the current barrier in one coordinate is u and its smaller directional eigenvalue is ℓ , a shift δ suffices when $u \leq 2\delta$ and $\delta \geq u - \ell$ (Lemma 2.5). The determinantal representation theorem [LPR05] expresses the relevant three-variable restriction as a symmetric matrix pencil. The rank-one matrix direction becomes a rank-two projection in this pencil. A Schur-complement comparison gives a lower bound on ℓ (Lemma 4.2), reducing the total shift required. Splitting each operator into many commuting copies turns these local shifts into a scalar schedule; Theorem 4.4 supplies an initial admissible root bound.

Step 3: Retain the directional moments. Polynomial padding permits the normalization $V = I$ (Lemma 4.6). Given an admissible root bound C , take $m > Cs$. At the scalar matrix mI , after processing variance $s^2 I$, put $q := s^2 / m^2$. For a unit direction v , define $\tau := (\sum_{i=1}^n (v^* B_i v)^2)^{1/2} \in [0, 1]$. In the normalized pencil, let P be the rank-two projection and T the coefficient of the variance variable. The spectral measure of T^2 weighted by $P/2$ is a probability measure; denote its variable by X . Lemmas 4.7, 4.10, and 4.12, together with the positive remainder in Lemma 4.18, give

$$\mathbb{E}[X] = q, \quad \mathbb{E}[X^2] \leq (2 - \tau^4)q^2, \quad \mathbb{E}[X^3] \leq 4q\mathbb{E}[X^2] - 3q^3.$$

Admissibility of C confines X to $[0, C^2 q] \subset [0, 1)$. The barrier is proportional to $\mathbb{E}[(1 - X)^{-1}]$. A cubic majorant for this resolvent combines the moment relations to give the directional barrier bound $U_\tau^{(6)}$ of Lemma 4.19. Retaining the relation between the fourth and sixth moments makes the resulting estimate sharper. Theorem 4.23 adds the coupled relation

$$\mathbb{E}[X^4] - 4q\mathbb{E}[X^3] + \frac{13}{4}q^2\mathbb{E}[X^2] \leq \frac{1}{2}q^4.$$

Its remainder is $Y - W^2$ plus a tensor Gram form with an explicit positive rational 20×20 coefficient matrix. A quartic majorant has coefficient $-4q$ times its leading coefficient, so substituting this coupled inequality cancels the third scalar moment exactly. The remaining second-moment coefficient is nonnegative. This gives $U_\tau^{(8)}$ in Lemma 4.25.

The tenth-moment improvement. Theorem 4.28 proves, for the same spectral variable X ,

$$25\mathbb{E}[X^5] - 141q\mathbb{E}[X^4] + 295q^2\mathbb{E}[X^3] - 295q^3\mathbb{E}[X^2] + 39q^5 \leq 0.$$

Its positive remainder consists of two elementary positive matrices and rational Gram blocks of sizes 120 and 5. A degree-five majorant matches the three leading coefficients of this coupled inequality, so the $\mathbb{E}[X^4]$ and $\mathbb{E}[X^3]$ terms cancel before the bound on $\mathbb{E}[X^2]$ is substituted. The resulting initial barrier has the form $U_z^{(10)} = U_0 - U_1 z^4$. The same variance-retaining compression and finite-copy argument then give seven exact improvements from 2.24710 to 2.22767, as detailed in Section 4.8.

The final twelfth-moment refinement. The directed moment expansion extends to order twelve, with 588 collected terms. A rational certificate expresses a coupled moment polynomial as 31 positive Gram blocks and a Hermitian residual. The polynomial constructions in Section A.5 produce the Gram matrices from short rational seed vectors. Every independently summed pair word has operator norm at most one, so the sum of the absolute residual coefficients bounds the residual norm. Adding 49/2500 to the scalar coefficient gives the twelfth-moment inequality in Theorem 4.34. A degree-six majorant cancels the fifth, fourth, and third scalar moments before the second-moment bound is substituted. This gives the initial barrier $U_z = U_0 - U_{\text{drop}}z^4$ in Lemma 4.37.

Step 4: Build the shift schedule. The same direction parameter improves the compression estimate. Starting at variance s_0^2 and scalar matrix m_0I , choose a shift rate $k(s)$ and put $m(s) := m_0 + \int_{s_0}^s 2uk(u) du$. Lemma 4.20 retains a positive Schur-complement correction and gives the normalized lower eigenvalue bound $\mathcal{L}(C; m, s, \tau)$ of Eq. (88). Theorems 4.21, 4.26, and 4.31 use this same compression estimate with the sixth-, eighth-, and tenth-moment initial barriers, respectively. For the final step, Lemma 4.35 obtains an additional constraint from the fourth-order coefficient of the directional Hessian. Theorem 4.36 uses it to improve the compressed inverse bound to $1/(m + \nu - \eta)$, with a rational correction η . The schedule uses this estimate for direction parameters $z \geq 1/4$ and the simple bound $1/(m + s/4)$ for $z \leq 1/4$. The uniform condition $m > 15s/8$ keeps all rational denominators positive. Theorem 4.38 accounts for the nonisotropic error during each partial round and maintains actual cone membership throughout the finite-copy induction.

Step 5: Certify the final constant. The rational schedules in Section 4 satisfy these inequalities by exact Bernstein coefficient checks on intervals and rectangles. Positivity of the Bernstein basis certifies the full domains. Fourth- and sixth-moment iterations improve the initial admissible bound; nine iterations of one directional schedule family reach 2.295145, and six schedules using both signs of the pencil reach 2.28669. The positive sixth-moment remainder and four further schedules reach 2.28128. The eighth-moment remainder and six schedules give the input 2.24710. The tenth-moment remainder and seven further schedules give 2.22767. The twelfth-moment inequality and Hessian compression then give three further schedules, with successive admissible bounds 2.20750, 2.20100, and 2.19868; see Section 4.11. The additional compression estimate of Lemma 4.39 and six further schedules in Section 4.11 give 2.19595. Finally, Section 4.12 applies a complete infinitesimal round to a fixed directional spectral measure. If its barrier is bounded by $g_v(a) \leq A - Bz_v^4$ and the local shifts are valid with rate K , then

$$\mathbb{E}\left[\frac{(a - K)Y - Ka^2}{(a^2 - Y)^2}\right] \leq 0.$$

Positive rational combinations of these resolvent inequalities and the coupled moment bounds give stronger initial barriers. The certificate establishes 20 resolvent constraints in order and two schedules, giving the admissible bounds 2.1812 and 2.17969. Section 4.13 adds another certified twelfth-moment inequality and six resolvent constraints; one further schedule gives 2.17502. The derivative inequality comes from an actual finite-round comparison, so no derivative of an upper bound is assumed. The analytic schedule lemma and these finite certificates establish Theorem 4.1.

3.5 Variance lower bounds from Gaussian moments

Step 1: Choose the Gaussian family. Theorem 5.1 gives, for every $0 < c < 2$, a finite real rank-one family for which every signing has norm greater than $c\sigma$. The construction uses a Gaussian matrix $G := [g_1 \cdots g_n] \in \mathbb{R}^{d \times n}$ with $n := \alpha d$ and sets $u_i := g_i/\sqrt{d}$, $A_i := u_i u_i^\top$. For $s \in \{-1, 1\}^n$, define $S_s := G \text{diag}(s) G^\top / d$ and $\text{Disc}(A) := \min_s \|S_s\|$. The proof combines a moment lower bound

valid for every fixed signing with concentration strong enough to make that bound simultaneous over all 2^n signings.

Step 2: Obtain the Catalan moment bound. Fix a positive integer k and let $\text{Cat}_k := \binom{2k}{k}/(k+1)$. Lemma 5.4 proves

$$\mathbb{E}[d^{-1} \text{tr}[S_s^{2k}]] \geq \text{Cat}_k \alpha^k \quad (s \in \{-1, 1\}^n). \quad (22)$$

The mechanism is a positivity property of the full Gaussian Wick expansion. After summing the indices constrained by one pairing, each even column class contributes n and each odd column class contributes $\sum_{i=1}^n s_i$. There are an even number of odd classes, so each complete pairing has a nonnegative contribution. Retaining the pairings indexed by noncrossing pairings of $2k$ positions gives Cat_k terms, each equal to α^k . This identifies the source of the constant 2: the Catalan growth rate satisfies $\text{Cat}_k^{1/(2k)} \rightarrow 2$.

Step 3: Extend the trace moments. To pass from expectation to all signings at once, restrict first to the common event $\mathcal{K} := \{\|G\| \leq R\sqrt{d}\}$, where $R := \sqrt{\alpha(1+\delta)}$ for a fixed $0 < \delta < 1$. Lemma 5.3 makes the complement exponentially unlikely when α is sufficiently large. On $\mathcal{K}$, the normalized trace moment $F_s(G) := d^{-1} \text{tr}[S_s^{2k}]$ is Lipschitz for Frobenius distance with constant $4kR^{4k-1}/d$. Extend it to the whole Gaussian space with the same Lipschitz constant and with values in $[0, R^{4k}]$. The expectation changes by $o(1)$ uniformly in s , because the complement of $\mathcal{K}$ has exponentially small probability and Gaussian moments grow only polynomially in d for fixed k and α .

Step 4: Control every signing simultaneously. Gaussian concentration now bounds the lower-tail probability for each extended moment by e^{-ad^2} for a constant $a > 0$ depending only on the fixed parameters. The union bound over all signings costs $2^n = e^{\alpha d \log 2}$, which is dominated by this quadratic exponent. The exceptional event $\mathcal{K}^c$ is common to all signings and is counted once. Lemma 5.5 consequently gives, with probability tending to one,

$$d^{-1} \text{tr}[S_s^{2k}] \geq \frac{1}{2} \text{Cat}_k \alpha^k \quad \text{for every } s \in \{-1, 1\}^n. \quad (23)$$

Since $d^{-1} \text{tr}[S_s^{2k}] \leq \|S_s\|^{2k}$, this implies $\text{Disc}(A) \geq (\text{Cat}_k/2)^{1/(2k)} \sqrt{\alpha}$.

Step 5: Compare discrepancy with variance. The variance scale is controlled on the same Gaussian family. The identity $\sum_{i=1}^n A_i^2 = \sum_{i=1}^n \|u_i\|^2 A_i$ bounds its norm by the product of the largest squared column norm and the covariance norm. Lemma 5.3 therefore gives $\sigma \leq (1+\delta)\sqrt{\alpha}$ with probability tending to one. Combining the two events yields a realization satisfying

$$\frac{\text{Disc}(A)}{\sigma} \geq \frac{(\text{Cat}_k/2)^{1/(2k)}}{1+\delta}. \quad (24)$$

For a prescribed $c < 2$, choose k large, then δ small, then a sufficiently large fixed integer α , and finally d large. The right side exceeds c and the required events have positive probability. This proves the lower coefficient 2 in the variance normalization. Applying covariance normalization to these realizations gives the isotropic lower bound of Section 3.2.

4 The discrepancy bound

Section 4.1 states the theorem and gives the determinantal representation. Section 4.2 establishes the initial admissible bound. Sections 4.3–4.6 sharpen the barrier and compression estimates using fourth and sixth directional moments. Sections 4.7–4.9 derive the higher-moment inequalities needed for the final refinement. Sections 4.10 and 4.11 combine these inequalities with refined

compression estimates and exact rational schedules. Sections 4.12 and 4.13 derive additional resolvent constraints and combine them with another twelfth-moment certificate to obtain the final bound. Section 4.14 transfers the resulting root bound to the original supports and proves the Lyapunov approximation. Appendix A collects the longer polynomial constructions and Gram-block specifications.

4.1 Main theorem and determinantal representation

The proof combines a rank-two compression with directional moments through order twelve. A polynomial shift schedule and an admissibility iteration, strengthened by resolvent constraints, give the universal bound 2.17502σ .

We reserve C_0 for the universal discrepancy constant. In each group of schedules, $\widehat{C}_j$ denotes its local sequence of admissible bounds, with the index restarted at zero for that group.

For an auxiliary family $D = (D_i)$ whose index set is not otherwise specified, write $D = (D_1, \dots, D_{N_D})$, where N_D is its cardinality. At each step of a finite-copy schedule, this enumeration refers to the currently processed family.

Theorem 4.1. *Let $A_i := u_i u_i^*$ with $u_i \in \mathbb{C}^d$, $i \in [n]$, and let ξ_i be independent real random variables with finite support. Put*

$$\mu_i := \mathbb{E}[\xi_i], \quad \tau_i^2 := \text{Var}[\xi_i], \quad \sigma^2 := \left\| \sum_{i=1}^n \tau_i^2 A_i^2 \right\|.$$

There are original support values $\varepsilon_i \in \text{supp}(\xi_i)$ such that

$$\left\| \sum_{i=1}^n (\varepsilon_i - \mu_i) A_i \right\| \leq c_* \sigma, \quad c_* := \frac{108751}{50000}.$$

When $\sigma = 0$, the discrepancy is zero.

The stated coefficient is $c_* = 2.17502$. The first part of the proof establishes an initial admissible bound; the fourth- and sixth-moment refinements give an intermediate bound. Retaining the direction parameter and then using both signs of the pencil give a further bootstrap bound. A positive sixth-moment remainder improves the initial barrier. Finally, positive eighth- and tenth-moment remainders give the input for a twelfth-moment inequality and a Hessian-based compression estimate. Three further rational schedules give 2.19868. A second Schur trial matrix improves small directions, giving 2.19595. The resolvent refinement in Section 4.12 gives 2.17969. Combining it with a further twelfth-moment certificate in Section 4.13 gives the stated value.

We use H_J and its cone Γ_J from Eq. (12). The determinantal input is the Helton–Vinnikov determinantal representation theorem in three real variables, equivalently the Lax theorem [LPR05]: a homogeneous hyperbolic polynomial $F(x, s, t)$ of degree q , normalized by $F(1, 0, 0) = 1$ and hyperbolic in the x direction, has a representation $F(x, s, t) = \det(xI_q + sA + tT)$ with real symmetric A, T . This result includes reducible polynomials. Only a three-variable restriction is represented; no determinantal representation of the full many-variable polynomial is assumed.

4.2 Compression estimates and the initial bound

We combine a rank-two compression estimate with a finite shift schedule to obtain the initial admissible bound.

Rank-two compression gives the directional eigenvalue bound needed for the local barrier update.

Lemma 4.2. *Let $(D_j)_{j \in J}$ be PSD rank-one matrices, $M \succ 0$, and $(M, 1) \in \Gamma_J$. Let $B \neq 0$ be PSD of rank one, and suppose $\sum_{j \in J} D_j^2 \preceq \theta I$. Put*

$$a := \operatorname{tr}[M^{-1}B], \quad y := \operatorname{tr}[M^{-2}B].$$

The polynomial $H_J(M + sB, 1)/H_J(M, 1)$ has two positive directional eigenvalues. Its smaller eigenvalue satisfies

$$\ell_B \geq \frac{a^2}{a + \sqrt{\theta} y}. \quad (25)$$

Proof. Write $B = bvv^*$ with $\|v\| = 1$ and $b > 0$. Under congruence by $M^{-1/2}$, define

$$C_j := M^{-1/2} D_j M^{-1/2}, \quad M^{-1/2} B M^{-1/2} = azz^*, \quad \|z\| = 1.$$

Normalize the ternary restriction by

$$F(x, s, t) := H_J(xM + sB, t) / \det(M)^2.$$

It is hyperbolic in $(1, 0, 0)$: $M \succ 0$ makes $(M, 0)$ an interior hyperbolicity direction. The determinantal representation theorem gives

$$F(x, s, t) = \det(xI_{2d} + sA + tT).$$

At $t = 0$, the determinant lemma implies $F(x, s, 0) = x^{2d-2}(x + as)^2$. Hence the symmetric matrix A has two eigenvalues equal to a and all others zero. Define $P := A/a$, which is an orthogonal projection of rank two.

We now determine the compression PTP on $\operatorname{ran} P$. Set $d_j := \operatorname{tr}[C_j]$ and $c_j := z^* C_j z$. For $x > 0$ and $x + as > 0$, put $\zeta := as/(x + as)$. The coefficient of t^2 in the original differential operator expression, divided by $F(x, s, 0)$, is

$$-x^{-2} \sum_{j \in J} (d_j - \zeta c_j)^2. \quad (26)$$

Indeed, each C_j has rank one, so $\frac{1}{2} \mathcal{D}_{C_j}^2 \det(X)^2 = \det(X)^2 \operatorname{tr}[X^{-1} C_j]^2$; products of two nonconstant operator terms have degree at least four in t .

Evenness of F in t first gives $\operatorname{tr}[T] = \operatorname{tr}[PT] = 0$. The determinant representation then gives its normalized t^2 coefficient as

$$-\frac{1}{2x^2} [\operatorname{tr}[T^2] - 2\zeta \operatorname{tr}[PT^2] + \zeta^2 \operatorname{tr}[PTPT]]. \quad (27)$$

Comparing the coefficients of ζ^2 in Eq. (26) and Eq. (27) yields

$$\operatorname{tr}[(PTP|_{\operatorname{ran} P})^2] = 2 \sum_{j \in J} c_j^2.$$

Together with its zero trace, this says that the two eigenvalues of $PTP|_{\operatorname{ran} P}$ are $\pm\nu$, where $\nu := (\sum_{j \in J} c_j^2)^{1/2}$.

Cone membership implies $I + T \succ 0$. One can see this along the segment (M, t) , $0 \leq t \leq 1$, which stays in Γ_J : the representing pencil starts at I and remains nonsingular, hence remains positive definite. The Schur complement inequality gives, on $\operatorname{ran} P$,

$$G := P(I + T)^{-1}P \succeq (I_P + PTP)^{-1} \succeq \frac{1}{1 + \nu} I_P.$$

If $\text{ran } P$ is the whole space, the first inequality is an equality. The two nonzero directional eigenvalues are the eigenvalues of aG , by the matrix determinant lemma. Thus $\ell_B \geq a/(1 + \nu) > 0$.

Finally, let $w := M^{-1}v$. Then

$$c_j = \frac{w^* D_j w}{v^* M^{-1} v}, \quad \sum_{j \in J} (w^* D_j w)^2 \leq \|w\|^2 w^* \left(\sum_{j \in J} D_j^2 \right) w \leq \theta \|w\|^4.$$

Consequently $\nu \leq \sqrt{\theta} y/a$, proving Eq. (25). This argument uses arbitrary-dimensional matrices; the dimension-two compression concerns only one rank-one direction inside a $2d$ -dimensional pencil. $\square$

An elementary resolvent comparison makes the compression estimate uniform over normalized variance matrices.

Lemma 4.3. *Let $M_0 \succeq hI$ with $h > 0$, let $s, \rho \geq 0$, and let $B \succeq 0$ be nonzero of rank one, with $b := \text{tr}[B]$. Define*

$$X_0 := \frac{\text{tr}[M_0^{-1}B]}{b}, \quad X := \frac{\text{tr}[(M_0 + sI)^{-1}B]}{b}, \quad Y := \frac{\text{tr}[(M_0 + sI)^{-2}B]}{b}.$$

Then

$$2X_0 - \frac{X^2}{X + \rho Y} \leq \frac{2}{h} - \frac{1}{h + s + \rho}. \quad (28)$$

Proof. In an eigenbasis of M_0 , these are weighted averages of $1/x$, $1/(x + s)$, and $1/(x + s)^2$, respectively, with $x \geq h$. Let $X_* := 1/(h + s)$ and $Y_* := X_*^2$. Then $X_* \geq X$, $Y_* \geq Y$, and

$$X_* - X \leq 1/h - X_0,$$

since this holds term by term in the weighted average. For $L(X, Y) := X^2/(X + \rho Y)$,

$$0 \leq \partial_X L = 1 - \left(\frac{\rho Y}{X + \rho Y} \right)^2 \leq 1, \quad \partial_Y L \leq 0.$$

Increasing first X and then Y gives $L(X_*, Y_*) - L(X, Y) \leq X_* - X \leq 1/h - X_0$. Since $L(X_*, Y_*) = 1/(h + s + \rho)$, the claimed inequality follows. $\square$

The local barrier update and compression estimate reduce admissibility to a finite set of scalar shift inequalities.

Theorem 4.4. *Choose $N \geq 1$, $h > 0$, and $k_r \geq 1/h$. Write*

$$S_r := \frac{1}{N} \sum_{j=1}^r k_j, \quad m_r := h + S_r.$$

If, for every $1 \leq r \leq N$,

$$k_r \geq \frac{2}{h} - \frac{1}{m_r + \sqrt{r/N}}, \quad (29)$$

then $h + S_N$ is admissible in the sense of Eq. (9). No previously admissible input constant is required.

Proof. Normalize $V = \sum_{i=1}^n B_i^2 \preceq I$, discard zero matrices, and put $b_i := \text{tr}[B_i]$. Use the N copies $B_i/\sqrt{N}$, shifts $\delta_{r,i} := k_r b_i/\sqrt{N}$, and starting matrix $M_0 := (h + S_N)I - S_N V \succeq hI$, at $x = h + S_N$ with all copy coordinates initially $-\delta_{r,i}$. The starting squared determinant is

$$Q(x, z) := \det(xI + \sum_{r=1}^N \sum_{i=1}^n z_{r,i} B_i/\sqrt{N})^2.$$

At this initial point its matrix is M_0 . Every nonnegative coordinate shift adds a PSD matrix, so the point belongs to Ab_Q . The initial unused barrier for a copy of B_i is $U_i := 2 \text{tr}[M_0^{-1} B_i]/\sqrt{N}$.

Before an update in round r , let J be the processed copies and M the current matrix. The induction gives $(M, 1) \in \Gamma_J$, while

$$M \preceq M_0 + S_r V \preceq K := M_0 + S_r I, \quad \sum_{j \in J} D_j^2 \preceq (r/N)I.$$

Adding the PSD direction $(K - M, 0)$ stays in the cone. The change-of-base Lemma 2.6 shows that each directional eigenvalue at $(M, 1)$ is at least the corresponding one at $(K, 1)$. Apply Lemma 4.2 at K . With X, Y from Lemma 4.3, $s = S_r$, and $\rho = \sqrt{r/N}$, the smaller eigenvalue in the copy direction is at least

$$\ell \geq \frac{b_i}{\sqrt{N}} \frac{X^2}{X + \rho Y}.$$

The maintained bound $u \leq U_i$ and Lemma 4.3 therefore give

$$u - \ell \leq \frac{b_i}{\sqrt{N}} \left(\frac{2}{h} - \frac{1}{m_r + \sqrt{r/N}} \right) \leq \delta_{r,i}.$$

Also $u \leq 2b_i/(h\sqrt{N}) \leq 2\delta_{r,i}$. Thus Lemma 2.5 preserves above-root membership and all unused barriers. This proves the induction.

At the end all copy coordinates are zero. The commuting-operator identity

$$\prod_{i=1}^n \left(1 - \frac{1}{2N} \mathcal{D}_{B_i}^2\right)^N \det(X)^2 = \prod_{i=1}^n \left(1 - \frac{1}{2} \mathcal{D}_{B_i}^2\right) \det(X)^2$$

follows from $\mathcal{D}_{B_i}^4 \det(X)^2 = 0$. All these directional derivatives commute, so the higher binomial terms vanish even after applying the operators belonging to other indices. Thus the final polynomial is the original P_B , proving the root bound. No support choices for the artificial copies are used. $\square$

A continuous schedule with a positive margin produces finite schedules satisfying Theorem 4.4.

Lemma 4.5. *Let k be continuous on $[0, 1]$ and put*

$$m(s) := h + \int_0^s 2uk(u) \, du, \quad g(s) := 1 + (k(s) - 2/h)(m(s) + s).$$

If $k(s) \geq 1/h$ and $g(s) \geq \gamma > 0$ throughout $[0, 1]$, then $m(1)$ is admissible.

Proof. For each N , take $k_r := k(\sqrt{r/N})$. Uniform convergence of right Riemann sums for the continuous function $t \mapsto k(\sqrt{t})$ gives

$$\max_{r \in [N]} \left| h + \frac{1}{N} \sum_{j=1}^r k(\sqrt{j/N}) - m(\sqrt{r/N}) \right| \longrightarrow 0.$$

The function $k - 2/h$ is bounded. Hence, for all sufficiently large N ,

$$1 + (k_r - 2/h)(m_r + \sqrt{r/N}) \geq \gamma/2 > 0.$$

The positive denominator makes this exactly the required inequality Eq. (29). The finite theorem applies, and its endpoints converge to $m(1)$. For each fixed original family, its largest root is at most every endpoint in this sequence, hence at most their limit. This proves admissibility without a floating-point limit argument. $\square$

Take $h := 1667/1250$ and

$$k(s) := 10^{-10} \sum_{i=0}^{10} a_i s^i,$$

where $a_i := [s^i]P^{(0)}(s)$ and

$$\begin{aligned} P^{(0)}(s) := & 7498500486 + 5622739300s + 367860s^2 \\ & - 1058940805s^3 - 752730513s^4 + 785829766s^5 \\ & + 490789016s^6 - 794023964s^7 + 271991836s^8 \\ & + 26934395s^9 - 24022430s^{10}. \end{aligned} \tag{30}$$

Define the exact rational polynomials

$$m(s) := h + 10^{-10} \sum_{i=0}^{10} \frac{2a_i}{i+2} s^{i+2}, \quad g(s) := 1 + (k(s) - 2/h)(m(s) + s).$$

Exact rational calculation gives

$$k(s) - 1/h \geq 10^{-8}, \quad g(s) \geq \frac{1}{250000000} \quad (0 \leq s \leq 1). \tag{31}$$

Here is the precise finite certificate and why it covers the entire interval. On each of the sixteen intervals $[j/16, (j+1)/16]$, substitute $s = j/16 + u/16$, and write a polynomial $p(s)$ of degree n as

$$p(j/16 + u/16) = \sum_{i=0}^n \beta_i \binom{n}{i} u^i (1-u)^{n-i}.$$

If c_i are its power coefficients after substitution, then

$$\beta_i = \sum_{r=0}^i c_r \frac{\binom{i}{r}}{\binom{n}{r}}.$$

All these computations use rational arithmetic. The Bernstein basis is nonnegative and sums to one on $[0, 1]$, so a lower bound on all β_i is a lower bound on the polynomial throughout that interval. This calculation evaluates $16 \cdot 11 = 176$ coefficients for $k - 1/h$ and $16 \cdot 23 = 368$ coefficients for g . Their respective minima are

$$\frac{155081}{833500000000} > 10^{-8}, \quad \frac{446874316811014577}{962692500000000000000000000} > \frac{1}{250000000}.$$

Thus Eq. (31) is an interval proof, not a grid test. The endpoint and its strict margin are exactly

$$c_{\text{init}} := m(1) = \frac{41813459322619}{173250000000000}, \quad \frac{24134753}{10000000} - c_{\text{init}} = \frac{249881}{173250000000000} > 0.$$

Lemma 4.5 now proves that c_{init} is admissible.

4.3 Fourth-moment bounds

We pad the polynomial to isotropy and use its fourth directional moment to sharpen the initial barrier.

The preceding argument supplies the admissible starting constant

$$C_{\text{in},4} := \frac{24134753}{10000000}.$$

Polynomial padding reduces the fourth-moment refinement to the isotropic case.

Lemma 4.6. *To prove admissibility of a constant, it suffices to consider families with $\sum_{i=1}^n B_i^2 = I$.*

Proof. Normalize $V \preceq I$. A spectral decomposition of $I - V$ provides PSD rank-one matrices $D_1, \dots, D_p$ satisfying $\sum_{j=1}^p D_j^2 = I - V$. Adding these matrices cannot decrease the largest root. To check this one matrix at a time, let H be the already processed homogeneous polynomial, let r be the largest root of $H(xI, 1)$, and put $\tilde{H} := H - \mathcal{D}_D^2 H/2$ at $t = 1$. For $x > r$, $(xI, 1)$ belongs to the hyperbolicity cone of H , and $(D, 0)$ belongs to its closure. Its directional eigenvalues are nonnegative, giving $\mathcal{D}_D^2 H(xI, 1) \geq 0$. Continuity gives $\tilde{H}(rI, 1) \leq 0$. The monic real-rooted polynomial $\tilde{H}(xI, 1)$ is positive for large x , and hence has a root at least r . Iterating proves the claim. The zero-variance case is immediate. $\square$

We next compute the second and fourth directional moments. For a family $D_j := w_j w_j^*$, $j \in [m]$, define

$$V_D := \sum_{j=1}^m D_j^2, \quad W_D := \sum_{i=1}^m \sum_{j=1}^m \text{tr}[D_i D_j] D_i D_j.$$

This last matrix is PSD. Indeed, with Z having columns w_j and $G := Z^* Z$, it equals $Z(G \circ G \circ \overline{G}) Z^*$, where $\circ$ denotes entrywise multiplication. The Schur product theorem applies because G and $\overline{G}$ are PSD.

The second and fourth moments of the representing pencil provide the additional information needed to sharpen the barrier estimate.

Lemma 4.7. *Let $B := bvv^*$ with $b > 0$ and $\|v\| = 1$. Write the ternary representation*

$$H_D(xI + sB, t) = \det(xI_{2d} + sbP + tT),$$

where P is a rank-two orthogonal projection. Then

$$\text{tr}[PT^{2j+1}] = 0 \quad (j \geq 0), \tag{32}$$

$$\text{tr}[PT^2] = 2v^* V_D v, \tag{33}$$

$$\text{tr}[PT^4] = 2v^* [V_D^2 + \sum_{i=1}^m \text{tr}[D_i V_D] D_i - W_D] v. \tag{34}$$

In particular, if $V_D = qI$, then

$$\frac{1}{2} \text{tr}[PT^2] = q, \quad \frac{1}{2} \text{tr}[PT^4] \leq 2q^2.$$

Proof. The representation and the form bP follow exactly as in Lemma 4.2. Differentiation in s at zero gives

$$\partial_s \log H_D(I + sB, t)|_{s=0} = b \text{tr}[P(I + tT)^{-1}].$$

The left side is even in t , proving Eq. (32) by its power series at zero.

Here is the coefficient calculation through order four. Put $L_s := I + sB$, $d_i(s) := \text{tr}[L_s^{-1}D_i]$, and $r_{ij}(s) := \text{tr}[L_s^{-1}D_i L_s^{-1}D_j]$. Rank one and the determinant lemma give

$$\frac{H_D(L_s, t)}{\det(L_s)^2} = 1 - t^2 E_1(s) + t^4 E_2(s) + O(t^6),$$

where

$$E_1(s) := \sum_{i=1}^m d_i(s)^2, \quad E_2(s) := \sum_{i=1}^{m-1} \sum_{j=i+1}^m (d_i(s)d_j(s) - r_{ij}(s))^2.$$

The coefficient for a pair follows because each determinant is affine in its two rank-one variables: differentiating their product twice in each variable selects the square of the two-variable determinant coefficient. At zero, write $d_i := \text{tr}[D_i]$, $r_{ij} := \text{tr}[D_i D_j]$ and $e_i := v^* D_i v$. Since $D_i^2 = d_i D_i$,

$$\begin{aligned} E_1(0) &= \text{tr}[V_D], & E_1'(0) &= -2b v^* V_D v, \\ E_2'(0) &= -2b \sum_{i=1}^{m-1} \sum_{j=i+1}^m (d_i d_j - r_{ij}) v^* [d_j D_i + d_i D_j - D_i D_j - D_j D_i] v. \end{aligned}$$

The coefficients of t^2, t^4 in the logarithmic derivative are $-E_1'(0)$ and $E_2'(0) - E_1(0)E_1'(0)$, respectively. The matrix sum in the last display simplifies to

$$(\text{tr}[V_D])V_D - V_D^2 - \sum_{i=1}^m \text{tr}[D_i V_D] D_i + W_D.$$

For clarity, the $d_i d_j$ terms give $(\text{tr}[V_D])V_D - V_D^2$; the r_{ij} terms give $-\sum_{i=1}^m \text{tr}[D_i V_D] D_i + W_D$, with the diagonal D_i^4 terms cancelling in both groups. This proves Eq. (33) and Eq. (34). If $V_D = qI$, the first two positive terms are both $q^2 I$; $W_D \succeq 0$ proves the final bound. $\square$

The moment identities give the following initial barrier. For $C > \sqrt{2}$, $0 < s < 1$, and $m > Cs$, define

$$q := \frac{s^2}{m^2}, \quad r := C^2 q, \quad a := q \frac{C^2 - 2}{C^2 - 1}, \quad U(C; m, s) := \frac{2}{m} \frac{1 - a - r + q}{(1 - a)(1 - r)}. \quad (35)$$

All denominators are positive.

The moment identities turn an admissible root bound into an upper bound for the initial directional barriers.

Lemma 4.8. *Suppose C is admissible and $\sum_{i=1}^n B_i^2 = I$. Put*

$$H_{sB}(X, 1) := \prod_{i=1}^n (1 - s^2 \mathcal{D}_{B_i}^2 / 2) \det(X)^2.$$

Then the point $(mI, 1)$ belongs to its cone, and every PSD rank-one direction B satisfies

$$\mathcal{D}_B \log H_{sB}(mI, 1) \leq (\text{tr}[B])U(C; m, s).$$

Proof. Admissibility gives the largest root at most $Cs < m$. In the normalized ternary representation, use $D_i := sB_i/m$, so $V_D = qI$. The direction matrix is $(\text{tr}[B])P/m$. The eigenvalues of T

are the negatives of the roots of $H_D(xI, 1)$. This polynomial is even in x , so $\|T\| \leq C\sqrt{q} < 1$. By Eq. (32),

$$\text{tr}[P(I + T)^{-1}] = \text{tr}[P(I - T^2)^{-1}].$$

The spectral measure of T^2 weighted by $P/2$ is a probability measure on $[0, r]$. If X denotes its scalar variable, the moment identities say $\mathbb{E}[X] = q$ and $\mathbb{E}[X^2] \leq 2q^2$.

We give the scalar majorant explicitly. For $0 \leq x \leq r < 1$, put

$$Q(x) := \frac{1}{1-a} + \frac{x-a}{(1-a)^2} + \frac{(x-a)^2}{(1-r)(1-a)^2}.$$

Direct subtraction gives

$$Q(x) - \frac{1}{1-x} = \frac{(r-x)(x-a)^2}{(1-r)(1-a)^2(1-x)} \geq 0.$$

The positive quadratic coefficient permits substitution of $\mathbb{E}[X^2] \leq 2q^2$. With $a = (rq - 2q^2)/(r - q)$, the result simplifies to

$$\mathbb{E}\left[\frac{1}{1-X}\right] \leq \frac{1-a-r+q}{(1-a)(1-r)}.$$

Multiplication by $2\text{tr}[B]/m$ proves the claimed barrier bound. $\square$

Starting from a partially processed polynomial allows the schedule to use the fourth-moment barrier estimate.

Lemma 4.9. *Suppose $C > \sqrt{2}$ is admissible, $0 < s_0 < 1$, $m_0 > Cs_0$, and $U := U(C; m_0, s_0)$. Let k be continuous on $[s_0, 1]$ and set*

$$m(s) := m_0 + \int_{s_0}^s 2uk(u) \, du.$$

If, throughout this interval,

$$k(s) \geq U/2, \quad 1 + (k(s) - U)(m(s) + s) \geq \gamma > 0, \quad (36)$$

then $m(1)$ is admissible.

Proof. By Lemma 4.6, assume $\sum_{i=1}^n B_i^2 = I$ and discard zero matrices. Put $b_i := \text{tr}[B_i]$, $\Delta := (1 - s_0^2)/N$, $t_r := s_0^2 + r\Delta$, $k_r := k(\sqrt{t_r})$, and $m_r := m_0 + \Delta \sum_{j=1}^r k_j$. The same uniform Riemann-sum argument as in Lemma 4.5 gives, for all sufficiently large N ,

$$k_r \geq U/2, \quad k_r \geq U - 1/(m_r + \sqrt{t_r}).$$

Start with the already processed polynomial $H_{s_0 B}$ and append N remaining copies $\sqrt{\Delta} B_i$ of each direction. Give each such copy the shift $\delta_{r,i} := k_r b_i \sqrt{\Delta}$. At $x = m_N$, with copy coordinates $-\delta_{r,i}$, the matrix argument is $m_0 I$, because $\sum_{i=1}^n b_i B_i = I$. Lemma 4.8 gives initial above-root membership and an unused barrier at most $b_i \sqrt{\Delta} U$.

Before an update in round r , the matrix argument satisfies $M \preceq m_r I$, and the sum of squares of all processed directions, including the initial $s_0 B_i$, is at most $t_r I$. Cone comparison and Lemma 4.2 therefore give

$$\ell \geq \frac{b_i \sqrt{\Delta}}{m_r + \sqrt{t_r}}.$$

The maintained barrier $u \leq b_i \sqrt{\Delta} U$ consequently satisfies $u \leq 2\delta_{r,i}$ and $u - \ell \leq \delta_{r,i}$. Lemma 2.5 applies at every update. At the end, for each i , the initial operator and all remaining operators combine to

$$(1 - \frac{s_0^2}{2} \mathcal{D}_{B_i}^2)(1 - \frac{\Delta}{2} \mathcal{D}_{B_i}^2)^N \det(X)^2 = (1 - \frac{1}{2} \mathcal{D}_{B_i}^2) \det(X)^2,$$

including after the other commuting operators, since $\mathcal{D}_{B_i}^4 \det(X)^2 = 0$ and $s_0^2 + N\Delta = 1$. Thus the largest root of the original polynomial is at most m_N . Taking $N \rightarrow \infty$ proves the claim. $\square$

We certify the fourth-moment schedule with the following rational parameters. Set

$$c_4 := \frac{4758489}{2000000} = 2.3792445, \quad m_0 := \frac{199}{125}, \quad s_0 := \frac{68}{125}, \quad L := 1 - s_0, \quad U_* := U(c_4; m_0, s_0).$$

Here c_4 is initially just a rational parameter; its admissibility will follow from the iteration below, without assuming the conclusion. With $z := (s - s_0)/L$, take

$$k_*(s) := 10^{-12} \sum_{i=0}^7 a_i z^i,$$

where $a_i := [z^i]P^{(4)}(z)$ and

$$\begin{aligned} P^{(4)}(z) := & 1017176353297 + 210553452866z - 36887684149z^2 \\ & - 4357231313z^3 + 3312657685z^4 - 186999346z^5 \\ & - 257865548z^6 + 64325072z^7. \end{aligned} \quad (37)$$

The corresponding polynomial is explicitly

$$m_*(s) := m_0 + 10^{-12} \sum_{i=0}^7 2La_i \left(\frac{s_0 z^{i+1}}{i+1} + \frac{Lz^{i+2}}{i+2} \right).$$

Exact Bernstein checks on the sixteen intervals in $0 \leq z \leq 1$ give

$$k_*(s) - U_*/2 \geq \frac{1}{4}, \quad 1 + (k_*(s) - U_*)(m_*(s) + s) \geq 10^{-9}. \quad (38)$$

There are $16 \cdot 8 = 128$ coefficients in the first check and $16 \cdot 17 = 272$ in the second. The construction and interval-coverage argument are the same Bernstein conversion used above, with rational arithmetic throughout. The endpoint is exactly

$$E_* := m_*(1) = \frac{2602298604464842839}{1093750000000000000} < c_4, \quad c_4 - E_* = \frac{67410157161}{1093750000000000000}. \quad (39)$$

We now extend this single schedule to every admissible $C \in [c_4, C_{\text{in},4}]$. Write $\eta := U(C; m_0, s_0) - U_*$ and put

$$k_C := k_* + 2\eta, \quad m_C(s) := m_*(s) + 2\eta(s^2 - s_0^2).$$

We verify both that $\eta \geq 0$ and that it grows slowly with C . For fixed $q := s_0^2/m_0^2 = 4624/39601$, let

$$A := C^2 - 1, \quad b(C) := q(A + 1 - 1/A), \quad f(C) := \frac{1}{1 - q - q^2/(1 - b(C))}.$$

Algebra in Eq. (35) gives $U(C; m_0, s_0) = 2f(C)/m_0$, whence

$$U'(C) = \frac{4Cq^3}{m_0} \left(1 + \frac{1}{(C^2 - 1)^2} \right) \frac{f(C)^2}{(1 - b(C))^2} > 0. \quad (40)$$

On $[c_4, C_{\text{in},4}]$, both b and f increase. Bounding all increasing factors at $C_{\text{in},4}$ and the factor $1 + (C^2 - 1)^{-2}$ at c_4 gives the rational bound

$$U'(C) \leq \frac{4C_{\text{in},4}q^3}{m_0} \left(1 + \frac{1}{(c_4^2 - 1)^2}\right) \frac{f(C_{\text{in},4})^2}{(1 - b(C_{\text{in},4}))^2} < \frac{1}{8}.$$

Exact rational substitution verifies this strict inequality, as well as $m_0 > C_{\text{in},4}s_0$, $b(C_{\text{in},4}) < 1$, and positivity of all displayed denominators. Therefore $0 \leq \eta \leq (C - c_4)/8$.

The barrier condition follows from $k_C - U(C)/2 = (k_* - U_*/2) + 3\eta/2$. If $g_C := 1 + (k_C - U(C))(m_C + s)$ and $g_* := 1 + (k_* - U_*)(m_* + s)$, direct expansion yields

$$g_C - g_* = \eta[m_* + s + 2(s^2 - s_0^2)(k_* - U_*)] + 2\eta^2(s^2 - s_0^2) \geq 0.$$

Indeed, $k_* \geq 0$, $m_* \geq m_0$, and the certificate also checks $U_* < 3/2$, so the expression in brackets is at least

$$m_0 + s_0 - 3(1 - s_0^2) = \frac{372}{15625} > 0.$$

Lemma 4.9 now makes $m_C(1)$ admissible whenever C is. Moreover,

$$m_C(1) = E_* + 2\eta(1 - s_0^2) < c_4 + \frac{1}{4}(C - c_4).$$

Starting from the already proved $C_{\text{in},4}$, induction shows that $c_4 + 4^{-j}(C_{\text{in},4} - c_4)$ is admissible for every integer $j \geq 0$. For each fixed family, its largest root is bounded by all these constants times $\|V\|^{1/2}$. Taking their limit proves that c_4 is admissible.

4.4 Sixth-moment bounds

The fourth-moment argument supplies the admissible constant c_4 . Throughout the moment estimates, μ_{2k} denotes the directional moment of order $2k$. We next exploit a relation between the fourth and sixth moments. The relation is stronger than bounding the two moments separately.

Lemma 4.10. *Suppose $D_i = w_i w_i^*$, $i \in [N]$, and $\sum_{i=1}^N D_i^2 = qI$, where $q > 0$. In the ternary representation of Lemma 4.7, define $\mu_{2j} := \text{tr}[PT^{2j}]/2$. Then*

$$\mu_2 = q, \quad \mu_4 \leq 2q^2, \quad \mu_6 \leq 3q\mu_4 - q^3 \leq 5q^3. \quad (41)$$

Proof. First take $q = 1$. Write $Z = [w_1 \ \cdots \ w_N]$, $G := Z^*Z$, $H := G \circ \bar{G}$, $K := G \circ G$, and

$$W := W_D = Z(G \circ H)Z^*, \quad \Psi(R) := \sum_{i=1}^N \text{tr}[D_i R] D_i.$$

Define two more PSD matrices

$$X := Z(\bar{G} \circ K^2)Z^*, \quad Y := Z(G \circ H^2)Z^*.$$

We will establish the exact identity

$$\mu_6 = v^*[5I - 5W - \Psi(W) + X + Y]v. \quad (42)$$

Here is its full determinant expansion. For $S \subseteq [N]$, let G_S be the principal Gram matrix and let P_S be the orthogonal projection onto $\text{span}\{w_i : i \in S\}$. Put

$$e_k := \sum_{\substack{S \subseteq [N] \\ |S|=k}} \det(G_S)^2, \quad M_k := \sum_{\substack{S \subseteq [N] \\ |S|=k}} \det(G_S)^2 P_S.$$

A summand with singular G_S is zero; equivalently its matrix part is $\det(G_S)Z_S \operatorname{adj}(G_S)Z_S^*$, so the formulas also hold at singular Gram matrices. As in the fourth-moment calculation,

$$H_D(I, t) = 1 - e_1 t^2 + e_2 t^4 - e_3 t^6 + O(t^8).$$

For $L_s := I + sbvv^*$, the coefficients of $H_D(L_s, t)/\det(L_s)^2$ are

$$e_k(s) := \sum_{\substack{S \subseteq [N] \\ |S|=k}} \det(Z_S^* L_s^{-1} Z_S)^2.$$

Their derivatives satisfy $e'_k(0) = -2bv^* M_k v$. Expanding the logarithmic derivative therefore gives

$$\mu_6 = v^*[(e_1^2 - e_2)M_1 - e_1 M_2 + M_3]v. \quad (43)$$

The lower-order identities imply

$$e_1 = d, \quad M_1 = I, \quad M_2 = (d-2)I + W, \quad e_2 = (d^2 - 2d + \operatorname{tr}[W])/2.$$

For the last equality use $\operatorname{tr}[M_2] = 2e_2$.

For completeness, the following finite expansion evaluates M_3 . For each ordered triple $\mathbf{i} = (i_1, i_2, i_3) \in [N]^3$, each $\pi, \sigma \in S_3$, and each $f \in [3]$, sum the terms

$$\frac{\operatorname{sgn}(\pi) \operatorname{sgn}(\sigma)}{6} \prod_{a=1}^3 G_{i_a i_{\pi(a)}} \prod_{a \neq f} G_{i_a i_{\sigma(a)}} w_{i_{\sigma(f)}} w_{i_f}^*. \quad (44)$$

This is the expansion of determinant times adjugate. Triples with repeated indices cancel because their determinant is zero; division by six changes ordered triples to subsets. Group the permutations into the identity E , the three transpositions T , and the two three-cycles R . Including their signs and the factor $1/6$, let S_{AB} denote the contribution from $\pi \in A$ and $\sigma \in B$. Then

$$\begin{aligned} S_{EE} &= d^2 I/2, & S_{ET} &= S_{TE} = -3dI/2, & S_{ER} &= S_{RE} = I, \\ S_{TT} &= (3 + \operatorname{tr}[W]/2)I + dW, & S_{TR} &= -3W, \\ S_{RT} &= -2W - \Psi(W), & S_{RR} &= X + Y. \end{aligned}$$

These reductions use only

$$\sum_{i=1}^N \|w_i\|^2 w_i w_i^* = I, \quad \sum_{i=1}^N \|w_i\|^2 G_{ji} G_{ik} = G_{jk}, \quad \sum_{i=1}^N \|w_i\|^4 = d.$$

Here are the nontrivial groups explicitly. Equal transpositions give $dW + \operatorname{tr}[W]I/2$, according as f is moved or fixed; distinct transpositions give $3I$. A transposition followed by a three-cycle gives $-3W$. Reversing these types gives $-2W$ when f is moved by the transposition and $-\Psi(W)$ when it is fixed. For two three-cycles, equal cycles give X and opposite cycles give Y : their respective sums, after relabeling, are

$$\begin{aligned} X &= \sum_{i,j,k=1}^N G_{ij} G_{jk}^2 G_{ki}^2 w_j w_i^*, \\ Y &= \sum_{i,j,k=1}^N G_{ki} |G_{ij}|^2 |G_{jk}|^2 w_k w_i^*. \end{aligned}$$

Summing the nine class contributions yields

$$M_3 = (d^2/2 - 3d + 5 + \text{tr}[W]/2)I + (d - 5)W - \Psi(W) + X + Y.$$

Substituting into Eq. (43) proves Eq. (42).

We now bound the positive terms. Both H and K are Gram matrices, and $0 \preceq H, K \preceq I$. To prove the upper bounds, their nonzero eigenvalues equal those of the frame operators of $w_i \otimes \bar{w}_i$ and $w_i \otimes w_i$, respectively. For every complex matrix R , Cauchy–Schwarz gives

$$\begin{aligned} \sum_{i=1}^N |w_i^* R w_i|^2 &\leq \sum_{i=1}^N \|w_i\|^2 \|R w_i\|^2 = \|R\|_F^2, \\ \sum_{i=1}^N |w_i^* R \bar{w}_i|^2 &\leq \sum_{i=1}^N \|w_i\|^2 \|R \bar{w}_i\|^2 = \|R\|_F^2. \end{aligned}$$

Thus $H^2 \preceq H$ and $K^2 \preceq K$. The Schur product theorem implies $0 \preceq X, Y \preceq W$. Also $\Psi(W) \succeq 0$. Writing $w := v^* W v \geq 0$ and using the exact fourth-moment identity, we obtain

$$\mu_4 = 2 - w, \quad \mu_6 \leq 5 - 3w = 3\mu_4 - 1.$$

Finally replace D_i by $D_i/\sqrt{q}$. This replaces T by $T/\sqrt{q}$, so its $2j$ th moment scales by q^{-j} . Rescaling proves Eq. (41). $\square$

A cubic majorant converts the sixth-moment relation into a barrier bound. Let $C \geq 2$, $0 < s < 1$, and $m > Cs$. Define

$$q := \frac{s^2}{m^2}, \quad r := C^2 q, \quad a := q \frac{2C^2 - 5}{C^2 - 2}, \quad U_6(C; m, s) := \frac{2}{m} \left[1 + q \frac{1 - a - r + 2q}{(1 - a)(1 - r)} \right]. \quad (45)$$

Here $0 < a < r < 1$.

Lemma 4.11. *Suppose C is admissible, $\sum_{i=1}^n B_i^2 = I$, and*

$$1 - r - 2a + 3q \geq 0. \quad (46)$$

Then every PSD rank-one direction B satisfies

$$\mathcal{D}_B \log H_{sB}(mI, 1) \leq \text{tr}[B] U_6(C; m, s).$$

Proof. Use the same spectral probability measure of T^2 weighted by $P/2$ as in Lemma 4.8. Its variable X belongs to $[0, r]$, and Lemma 4.10 gives

$$\mathbb{E}[X] = q, \quad \mathbb{E}[X^2] \leq 2q^2, \quad \mathbb{E}[X^3] \leq 3q\mathbb{E}[X^2] - q^3.$$

With $D := (1 - r)(1 - a)^2$, put

$$Q_3(x) := 1 + c_0 x + c_1 x^2 + c_2 x^3,$$

where

$$c_0 := \frac{(1 - 2a)(1 - r) + a^2}{D}, \quad c_1 := \frac{1 - r - 2a}{D}, \quad c_2 := \frac{1}{D}.$$

Multiplying the quadratic majorant in the proof of Lemma 4.8 by x and adding one shows

$$Q_3(x) - \frac{1}{1-x} = \frac{x(r-x)(x-a)^2}{D(1-x)} \geq 0 \quad (0 \leq x \leq r).$$

Although c_1 need not be nonnegative, Eq. (46) gives $c_1 + 3qc_2 \geq 0$. Consequently

$$\begin{aligned} \mathbb{E}[Q_3(X)] &\leq 1 + c_0q + (c_1 + 3qc_2)\mathbb{E}[X^2] - c_2q^3 \\ &\leq 1 + c_0q + 2c_1q^2 + 5c_2q^3 \\ &= 1 + q \frac{1-a-r+2q}{(1-a)(1-r)}. \end{aligned}$$

The last equality uses the displayed choice of a . Multiplication by $2 \operatorname{tr}[B]/m$ proves the claim. $\square$

Lemma 4.9 also applies with $U_6(C; m_0, s_0)$ in place of $U(C; m_0, s_0)$, provided Eq. (46) holds at (m_0, s_0) . Indeed, its proof uses the initial barrier estimate only to establish $u \leq b_i \sqrt{\Delta} U$. Lemma 4.11 supplies that estimate, and every subsequent finite update and limiting argument is unchanged.

We now certify the sixth-moment schedule. Set

$$c_6 := \frac{232863}{100000} = 2.32863, \quad m_0 := \frac{221}{125}, \quad s_0 := \frac{353}{500}, \quad L := 1 - s_0, \quad U_* := U_6(c_6; m_0, s_0).$$

The symbol c_6 is a parameter until admissibility is proved below. For $z := (s - s_0)/L$, define

$$k_*(s) := 10^{-12} \sum_{i=0}^5 a_i z^i, \quad m_*(s) := m_0 + 10^{-12} \sum_{i=0}^5 2La_i \left(\frac{s_0 z^{i+1}}{i+1} + \frac{Lz^{i+2}}{i+2} \right),$$

where $a_i := [z^i]P^{(6)}(z)$ and

$$\begin{aligned} P^{(6)}(z) &:= 1060241270869 + 119943639371z - 16552279255z^2 \\ &\quad + 17304783z^3 + 435707701z^4 - 64941663z^5. \end{aligned} \tag{47}$$

Exact Bernstein coefficients on each of the sixteen intervals in $z \in [0, 1]$ establish the following inequalities:

$$k_*(s) - U_*/2 \geq \frac{1}{4}, \quad g_*(s) := 1 + (k_*(s) - U_*)(m_*(s) + s) \geq \frac{1}{2000000000}. \tag{48}$$

The degrees are five and twelve, so these require $16 \cdot 6 = 96$ and $16 \cdot 13 = 208$ coefficients. The exact endpoint and margin are

$$E_* := m_*(1) = \frac{2910782122210742901}{1250000000000000000}, \quad c_6 - E_* = \frac{5377789257099}{1250000000000000000} > 0.$$

We again extend one schedule over an interval of admissible constants, now $C \in [c_6, c_4]$. For the fixed value $q = 124609/781456$, define

$$b(C) := q(C^2 - \frac{1}{C^2 - 2}), \quad f(C) := \frac{1}{1 - 2q - q^2/(1 - b(C))}.$$

Eq. (45) becomes $U_6(C; m_0, s_0) = 2(1 + qf(C))/m_0$. Hence

$$U'_6(C) = \frac{4Cq^4}{m_0} \left(1 + \frac{1}{(C^2 - 2)^2} \right) \frac{f(C)^2}{(1 - b(C))^2} > 0.$$

The following uniform upper bound is a strict rational check:

$$U'_6(C) \leq \frac{4c_4q^4}{m_0} \left(1 + \frac{1}{(c_6^2 - 2)^2}\right) \frac{f(c_4)^2}{(1 - b(c_4))^2} < \frac{4}{5}. \quad (49)$$

Here b, f increase with C , whereas $1 + (C^2 - 2)^{-2}$ decreases. Exact rational substitution also verifies $m_0 > c_4s_0$, positivity of the denominators, and Eq. (46) at $C = c_4$. That last check covers the entire interval: $r = C^2q$ and $a = q(2 - 1/(C^2 - 2))$ both increase with C , so $1 - r - 2a + 3q$ decreases.

Put $\eta := U_6(C; m_0, s_0) - U_*$, $k_C := k_* + 2\eta$, and $m_C(s) := m_*(s) + 2\eta(s^2 - s_0^2)$. Then $0 \leq \eta \leq 4(C - c_6)/5$ and $k_C \geq U_6(C)/2$. The same residual expansion as in the fourth-moment iteration gives

$$g_C - g_* = \eta[m_* + s + 2(s^2 - s_0^2)(k_* - U_*)] + 2\eta^2(s^2 - s_0^2) \geq 0.$$

Indeed, $U_* < 3/2$, $k_* \geq 0$, and $m_* \geq m_0$; the bracket is at least $m_0 + s_0 - 3(1 - s_0^2) = 242327/250000 > 0$. Thus the schedule lemma makes $m_C(1)$ admissible whenever C is. Finally,

$$m_C(1) = E_* + 2\eta(1 - s_0^2) < c_6 + \frac{9}{10}(C - c_6),$$

because $2(1 - s_0^2)(4/5) = 125391/156250 < 9/10$. Starting from the previously proved admissible c_4 , induction gives admissibility of $c_6 + (9/10)^j(c_4 - c_6)$ for every $j \geq 0$. Taking the limit for each fixed family proves admissibility of c_6 .

The three intermediate schedules require 544 initial coefficients, 400 fourth-moment coefficients, and 304 sixth-moment coefficients, for a total of 1248 exact interval coefficient checks. Rational expansion and substitution verify the scalar majorant identity and the derivative, coefficient-sign, endpoint, and contraction conditions. The matrix moment identities and their dimension-independent deductions are proved above.

4.5 Directional barriers and compression estimates

The preceding argument proves the admissible constant $c_6 = 2.32863$. We now retain a directional parameter in both the moment estimate and the compression estimate. This gives a further improvement without requiring moments of order eight.

Lemma 4.12. *Suppose $B_i = w_i w_i^*$ and $\sum_{i=1}^n B_i^2 = I$. For a unit vector v , define*

$$\tau := \sqrt{\sum_{i=1}^n (v^* B_i v)^2}, \quad M_v := \sum_{i=1}^n (v^* B_i v) B_i.$$

With $W = W_B$ and Ψ as in the sixth-moment proof,

$$0 \leq \tau \leq 1, \quad v^* W v \geq \tau^4, \quad v^* \Psi(W) v \geq \tau^6. \quad (50)$$

Consequently, for the pencil associated with $D_i = \sqrt{q} B_i$ and $\mu_{2j} = \text{tr}[PT^{2j}]/2$,

$$\mu_2 = q, \quad \mu_4 \leq (2 - \tau^4)q^2, \quad \mu_6 \leq 3q\mu_4 - (1 + \tau^6)q^3. \quad (51)$$

Proof. The scalar Cauchy–Schwarz inequality gives $(v^* B_i v)^2 \leq v^* B_i^2 v$, so $\tau \leq 1$. Define a linear map into a tensor product by

$$Sx := \sum_{i=1}^n (w_i^* x) w_i \otimes w_i \otimes \bar{w}_i.$$

The Gram matrix of the tensor vectors has entries $G_{ij}^2 \overline{G}_{ij}$, hence $S^*S = W$. The unit tensor $h := v \otimes v \otimes \bar{v}$ satisfies $S^*h = M_v v$. It follows that

$$W = S^*S \succeq S^*h h^*S = M_v v v^* M_v.$$

Since $v^* M_v v = \tau^2$, this proves the first nontrivial bound in Eq. (50). Also

$$v^* \Psi(W) v = \text{tr}[M_v W] \geq v^* M_v^3 v \geq (v^* M_v v)^3 = \tau^6.$$

The second inequality follows by applying convexity of x^3 on $[0, \infty)$ to the spectral probability measure of the PSD matrix M_v at v . For $q = 1$, the exact moment identities and $X, Y \preceq W$ give

$$\mu_4 = 2 - v^* W v, \quad \mu_6 \leq 5 - 3v^* W v - v^* \Psi(W) v = 3\mu_4 - 1 - v^* \Psi(W) v.$$

Use Eq. (50), then rescale by q , to obtain Eq. (51). $\square$

We use the directional parameter in the initial barrier. Fix an admissible $C \geq 2$, $0 < s_0 < 1$, and $m_0 > C s_0$. For $x \in [0, 1]$, set

$$\begin{aligned} q &:= s_0^2 / m_0^2, & r &:= C^2 q, \\ d(x) &:= 2 - x^4, & e(x) &:= 5 - 3x^4 - x^6, \\ A(x) &:= C^2 - d(x), & B(x) &:= q[C^2 d(x) - e(x)], \\ a(x) &:= B(x)/A(x). \end{aligned}$$

Here $d(x)$ is a scalar polynomial, unrelated to the matrix dimension. Assume the following inequalities throughout $[0, 1]$:

$$0 < a(x) < r < 1, \quad 1 - r - 2a(x) + 3q \geq 0. \quad (52)$$

Define

$$D(x) := (1 - r)[A(x) - B(x)], \quad (53)$$

$$N(x) := \frac{2}{m_0} \{D(x) + q[(1 - r + d(x)q)A(x) - B(x)]\}, \quad (54)$$

$$U_C(x) := N(x)/D(x). \quad (55)$$

These are polynomials of degrees at most six and eight in the denominator and numerator, respectively, and $D(x) > 0$.

Lemma 4.13. *Suppose $\sum_{i=1}^n B_i^2 = I$ and Eq. (52) holds. For a PSD rank-one direction $B = b v v^*$ with $\|v\| = 1$, let τ be its parameter from Lemma 4.12. Then*

$$\mathcal{D}_B \log H_{s_0 B}(m_0 I, 1) \leq b U_C(\tau).$$

Proof. Use the spectral measure and the cubic majorant Q_3 from Lemma 4.11, now with $a = a(\tau)$. The majorant identity holds for every $a < 1$; its proof does not require the previous choice of a . The coefficient condition is precisely $c_1 + 3q c_2 \geq 0$. Lemma 4.12 therefore gives

$$\begin{aligned} \mathbb{E}[Q_3(X)] &\leq 1 + c_0 q + (c_1 + 3q c_2) \mathbb{E}[X^2] - c_2 (1 + \tau^6) q^3 \\ &\leq 1 + c_0 q + c_1 d(\tau) q^2 + c_2 e(\tau) q^3 \\ &= 1 + q \frac{1 - a - r + d(\tau) q}{(1 - a)(1 - r)}. \end{aligned}$$

The last equality follows by substituting $a = q[C^2 d(\tau) - e(\tau)]/[C^2 - d(\tau)]$; equivalently, $a(rq - d(\tau)q^2) = rd(\tau)q^2 - e(\tau)q^3$. Multiplying by $2b/m_0$ yields the stated rational expression $b U_C(\tau)$. $\square$

Lemma 4.14. *Under the preceding hypotheses, let k be continuous on $[s_0, 1]$ and $m(s) := m_0 + \int_{s_0}^s 2uk(u) du$. Suppose, for every $(s, x) \in [s_0, 1] \times [0, 1]$,*

$$2k(s) \geq U_C(x), \quad 1 + [k(s) - U_C(x)][m(s) + sx] \geq \gamma > 0. \quad (56)$$

Then $m(1)$ is admissible.

Proof. Use isotropic padding and the finite-copy construction from Lemma 4.9. For each nonzero $B_i = b_i v_i v_i^*$, keep the fixed parameter $\tau_i := \sqrt{\sum_{j=1}^n (v_i^* B_j v_i)^2} \in [0, 1]$. Lemma 4.13 initializes its unused-copy barrier at most $b_i \sqrt{\Delta} U_C(\tau_i)$. Barrier monotonicity maintains this bound through the subsequent updates.

The smaller eigenvalue estimate also retains τ_i . At the scalar comparison point $m_r I$, the compression parameter in the proof of Lemma 4.2 is

$$\nu_i = \frac{1}{m_r} \sqrt{\sum_{j=1}^{N_D} (v_i^* D_j v_i)^2} \leq \frac{\sqrt{t_r} \tau_i}{m_r}.$$

Indeed, each original direction has total processed squared weight at most t_r , including its initial weight s_0^2 . The proof of that lemma gives the bound $a/(1 + \nu_i)$ before replacing ν_i by its direction-independent upper bound. Cone comparison from $M \preceq m_r I$ thus gives

$$\ell_i \geq \frac{b_i \sqrt{\Delta}}{m_r + \sqrt{t_r} \tau_i}.$$

Uniform continuity and the strict margin in Eq. (56) imply, for all sufficiently large numbers of rounds,

$$k_r \geq U_C(\tau_i)/2, \quad k_r \geq U_C(\tau_i) - \frac{1}{m_r + \sqrt{t_r} \tau_i} \quad \text{for every } r, i.$$

Consequently the shifts $\delta_{r,i} = k_r b_i \sqrt{\Delta}$ satisfy $u_i \leq 2\delta_{r,i}$ and $u_i - \ell_i \leq \delta_{r,i}$. The local barrier update applies. Combining the copies and taking the Riemann-sum limit proves admissibility of $m(1)$, exactly as before. $\square$

One polynomial family supplies the directional schedules. Set $C_* := 459029/200000$, $C_{\text{in,dir}} := 232863/100000$, $m_0 := 869/500$, and $s_0 := 1403/2000$. Define

$$p(z) := 10^{-8}(103362080 + 12915094z - 1300789z^2 - 158036z^3 + 71360z^4 - 8789z^5). \quad (57)$$

For $C \in [C_*, C_{\text{in,dir}}]$ and $s \in [s_0, 1]$, put $z := (s - s_0)/(1 - s_0)$ and define

$$k_C(s) := p(z) + \frac{C - C_*}{2}, \quad (58)$$

$$m_C(s) := m_0 + \int_{s_0}^s 2uk_C(u) du. \quad (59)$$

The endpoint is affine in C :

$$F(C) := m_C(1) = E + \beta(C - C_*), \quad (60)$$

where

$$E := \frac{64264053332016323}{28000000000000000}, \quad (61)$$

$$\beta := \frac{2031591}{8000000}. \quad (62)$$

The following estimate permits the same polynomial family at every stage.

Lemma 4.15. *For every admissible $C \in [C_*, C_{\text{in,dir}}]$, the value $F(C)$ is admissible. In particular, $C_* = 459029/200000$ is admissible.*

Proof. The inequalities $0 < s_0 < 1$ and $m_0 > C_{\text{in,dir}}s_0$ hold exactly. Use A, B, q, r from the setup of Lemma 4.13, and D, N from Eqs. (53)–(54). The required domain inequalities follow from $r < 1$ and

$$\min\{A, B, rA - B, (1 - r + 3q)A - 2B, D\} \geq 10^{-6}. \quad (63)$$

We also verify the following polynomial inequalities throughout $C \in [C_*, C_{\text{in,dir}}]$ and $(z, x) \in [0, 1]^2$:

$$(2k_C - 1/10)D - N \geq 0, \quad (64)$$

$$(1 - 10^{-8})D + (k_C D - N)(m_C + sx) \geq 0. \quad (65)$$

Here is the exact Bernstein verification of these inequalities. For each of the five domain polynomials in Eq. (63), subtract 10^{-6} . Together with the two left-hand sides in Eqs. (64)–(65), this gives seven polynomials to certify. Substitute $C = C_* + (C_{\text{in,dir}} - C_*)t$. For each resulting polynomial $H(t, z, x) = \sum_{i=0}^d h_i(z, x)t^i$, its Bernstein coefficients in t are

$$\widehat{H}_j(z, x) := \sum_{i=0}^j h_i(z, x) \frac{\binom{j}{i}}{\binom{d}{i}}, \quad 0 \leq j \leq d.$$

It suffices to prove that every $\widehat{H}_j$ is nonnegative on the unit square, since the Bernstein basis in t is nonnegative.

For a polynomial $P(z, x) = \sum_{i=0}^n \sum_{j=0}^m p_{ij} z^i x^j$, its tensor Bernstein coefficients on the unit square are

$$b_{uv} = \sum_{i=0}^u \sum_{j=0}^v p_{ij} \frac{\binom{u}{i}}{\binom{n}{i}} \frac{\binom{v}{j}}{\binom{m}{j}}.$$

Nonnegative coefficients prove nonnegativity on the square. Otherwise, bisect the rectangle at its midpoint, starting with the z coordinate and alternating coordinates at successive depths. For a univariate polynomial, split only its nonconstant coordinate. Apply the same test to both children and accept only rectangles with nonnegative coefficients. For a one-dimensional Bernstein vector $(b_0, \dots, b_n)$, the left and right half-interval vectors, each multiplied by the common positive factor 2^n , are

$$b_i^L = 2^{n-i} \sum_{h=0}^i \binom{i}{h} b_h, \quad b_i^R = 2^i \sum_{h=i}^n \binom{n-i}{h-i} b_h.$$

Applying these formulas to rows or columns gives exact rectangle subdivision after clearing a positive common denominator.

For the five domain polynomials and the half-barrier polynomial, every parameter coefficient $\widehat{H}_j$ is certified without subdivision. The last polynomial has degree six in t and bidegree at most $(12, 9)$

in (z, x) ; all seven parameter coefficients are certified by depth twelve. Thus this finite rational calculation establishes the inequalities on the full parameter interval and square.

Since $D > 0$, division in Eqs. (64)–(65) gives

$$\begin{aligned} 2k_C(s) - U_C(x) &\geq 1/10, \\ 1 + [k_C(s) - U_C(x)][m_C(s) + sx] &\geq 10^{-8}. \end{aligned} \tag{66}$$

Lemma 4.14 therefore makes $F(C)$ admissible whenever C is admissible in the specified interval.

To obtain C_* , start with the admissible $\hat{C}_0 := C_{\text{in,dir}} = c_6$ and define $\hat{C}_{j+1} := F(\hat{C}_j)$. The affine recurrence gives

$$\hat{C}_j - C_* = \beta^j (C_{\text{in,dir}} - C_*) - (C_* - E) \frac{1 - \beta^j}{1 - \beta}.$$

Exact rational comparison yields $\hat{C}_8 > C_*$ and $\hat{C}_9 < C_* - 1/6000000 < C_*$. Since $0 < \beta < 1$ and $E < C_*$, the inputs $\hat{C}_0, \dots, \hat{C}_8$ decrease within $[C_*, C_{\text{in,dir}}]$. Nine applications of the first assertion prove admissibility of $\hat{C}_9$, and hence of C_* . $\square$

The preceding argument discarded the positive Schur-complement correction. Evenness of the ternary polynomial allows us to retain part of that correction without assuming a smaller root bound.

Lemma 4.16. *Let $(D_j)_{j \in J}$ be PSD rank-one matrices and suppose $(mI, 1) \in \Gamma_J$, where $m > 0$. For $B = bvv^*$ with $\|v\| = 1$ and $b > 0$, put*

$$q_v := v^* \left(\sum_{j \in J} D_j^2 \right) v, \quad \nu^2 := \sum_{j \in J} (v^* D_j v)^2, \quad \beta := q_v - \nu^2 \geq 0.$$

The smaller directional eigenvalue of $H_J(mI + yB, 1)/H_J(mI, 1)$ satisfies

$$\ell_B \geq \frac{b}{m - \beta/(2m) + \sqrt{\nu^2 + \beta^2/(4m^2)}}. \tag{67}$$

Proof. Use the representation $H_J(xI + yB, t) = \det(xI + byP + tT)$, with P a rank-two orthogonal projection. The earlier coefficient identities give, on its range,

$$A := PTP, \quad \text{tr}[A] = 0, \quad A^2 = \nu^2 I_P, \quad \text{tr}[PT^2] = 2q_v.$$

The scalar polynomial $\det(xI + T) = H_J(xI, 1)$ is even, so the spectrum of T is symmetric. Cone membership gives $mI + T \succ 0$; hence $\|T\| < m$ and also $mI - T \succ 0$. Relative to $\text{ran } P \oplus \ker P$, write

$$T = \begin{pmatrix} A & R \\ R^* & E \end{pmatrix}, \quad G_{\pm} := P(mI \pm T)^{-1}P|_{\text{ran } P}, \quad K_{\pm} := G_{\pm}^{-1}.$$

Here the restriction symbol only specifies the two-dimensional space. If the complementary space is zero-dimensional, omit R and E . The Schur complement and $\|E\| < m$ imply

$$K_{\pm} = mI_P \pm A - R(mI \pm E)^{-1}R^* \preceq mI_P \pm A - \frac{RR^*}{2m}.$$

Since $\text{tr}[RR^*] = 2\beta$, write $RR^* = \beta I_P + Q$, where $\text{tr}[Q] = 0$ and Q has eigenvalues $\pm e$ with $0 \leq e \leq \beta$. Evenness in t and the determinant lemma give the polynomial identity

$$\det(I_P + byG_+) = \frac{H_J(mI + yB, 1)}{H_J(mI, 1)} = \frac{H_J(mI + yB, -1)}{H_J(mI, -1)} = \det(I_P + byG_-).$$

Thus G_+ and G_- have the same two eigenvalues, as do K_+ and K_- . Put $h := 1/(2m)$ and let Λ be their common largest eigenvalue. The preceding matrix inequalities yield

$$\Lambda \leq m - h\beta + \min_{\varepsilon \in \{-1,1\}} \lambda_{\max}(\varepsilon A - hQ).$$

A traceless Hermitian two-by-two matrix L has eigenvalues $\pm \sqrt{\text{tr}[L^2]}/2$. Consequently

$$\begin{aligned} \min_{\varepsilon \in \{-1,1\}} \lambda_{\max}(\varepsilon A - hQ)^2 &= \nu^2 + h^2 e^2 - h |\text{tr}[AQ]| \\ &\leq \nu^2 + h^2 \beta^2. \end{aligned}$$

Both largest eigenvalues here are nonnegative, so $\Lambda \leq m - h\beta + \sqrt{\nu^2 + h^2 \beta^2}$. The smaller eigenvalue of bG_+ is b/Λ , proving the claim. $\square$

For $m > s \geq 0$ and $0 \leq x \leq 1$, define

$$a := \frac{s^2(1-x^2)}{2m}, \quad L(m, s, x) := m - a + \sqrt{s^2 x^2 + a^2}. \quad (68)$$

It satisfies $m \leq L(m, s, x) \leq m + sx$, with a strict second inequality when $s > 0$ and $0 < x < 1$. Thus the new lower eigenvalue bound $b/L(m, s, x)$ improves $b/(m + sx)$ in those directions.

Lemma 4.17. *Fix an admissible $C \geq 2$, $0 < s_0 < 1$, and $m_0 > Cs_0$ satisfying Eq. (52). Let U_C be the barrier function in Lemma 4.13. Suppose k is continuous on $[s_0, 1]$ and*

$$m(s) := m_0 + \int_{s_0}^s 2uk(u) du > s.$$

If, throughout $[s_0, 1] \times [0, 1]$,

$$2k(s) \geq U_C(x), \quad 1 + [k(s) - U_C(x)]L(m(s), s, x) \geq \gamma > 0, \quad (69)$$

then $m(1)$ is admissible.

Proof. Use the finite-copy construction of Lemma 4.14, with $\Delta = (1 - s_0^2)/N$, $t_r = s_0^2 + r\Delta$, $k_r = k(\sqrt{t_r})$, and $m_r = m_0 + \Delta \sum_{h=1}^r k_h$. At an update in round r , all preceding rounds have been completed. The processed variance matrix therefore satisfies

$$t_{r-1}I \preceq \sum_{j=1}^{N_D} D_j^2 \preceq t_r I.$$

For the fixed original direction v_i , its parameters in Lemma 4.16 obey

$$q_i \geq t_{r-1}, \quad \nu_i^2 \leq t_r \tau_i^2, \quad \beta_i \geq \max\{0, t_{r-1} - t_r \tau_i^2\} =: \beta_{r,i}.$$

The last bound uses both $\beta_i = q_i - \nu_i^2$ and $\beta_i \geq 0$. The current matrix is at most $m_r I$. Cone comparison and the new compression lemma give, for the copy $\sqrt{\Delta} B_i$,

$$\ell_i \geq \frac{b_i \sqrt{\Delta}}{L_{r,i}}, \quad L_{r,i} := m_r - \frac{\beta_{r,i}}{2m_r} + \sqrt{t_r \tau_i^2 + \frac{\beta_{r,i}^2}{4m_r^2}}.$$

Indeed, $m - \beta/(2m) + \sqrt{\nu^2 + \beta^2/(4m^2)}$ increases with $\nu \geq 0$ and decreases with $\beta \geq 0$. Uniform convergence of the Riemann sums gives $m_r - m(\sqrt{t_r}) \rightarrow 0$. Also $\beta_{r,i} - t_r(1 - \tau_i^2) \rightarrow 0$ uniformly, including $\tau_i = 0, 1$. Hence $L_{r,i} - L(m(\sqrt{t_r}), \sqrt{t_r}, \tau_i) \rightarrow 0$ uniformly. These denominators are positive, and the strict margin in Eq. (69) implies, for sufficiently large N ,

$$k_r \geq U_C(\tau_i) - 1/L_{r,i}, \quad 2k_r \geq U_C(\tau_i).$$

The maintained unused barrier satisfies $u_i \leq b_i \sqrt{\Delta} U_C(\tau_i)$. Thus the shift $\delta_{r,i} = k_r b_i \sqrt{\Delta}$ satisfies $u_i \leq 2\delta_{r,i}$ and $u_i - \ell_i \leq \delta_{r,i}$. The local update preserves cone membership and all unused barriers. This establishes the induction, combines the copies, and bounds the largest root by m_N . Taking $N \rightarrow \infty$ proves the assertion. $\square$

We next certify schedules using the compression estimate for both signs. Use the fixed parameters

$$m_0 := \frac{34713}{20000}, \quad s_0 := \frac{1407}{2000}, \quad L_0 := 1 - s_0, \quad z := (s - s_0)/L_0.$$

For $j = 0, \dots, 5$, let $a_{j,i} := [z^i]P_j^{\text{Sch}}(z)$ and set

$$k_j(s) := 10^{-12} \sum_{i=0}^5 a_{j,i} z^i,$$

$$m_j(s) := m_0 + 10^{-12} \sum_{i=0}^5 2L_0 a_{j,i} \left(\frac{s_0 z^{i+1}}{i+1} + \frac{L_0 z^{i+2}}{i+2} \right).$$

$$P_0^{\text{Sch}}(z) := 1031223583180 + 127527938659z - 10557070853z^2 - 1896832389z^3 + 608775589z^4 - 60016716z^5. \quad (70)$$

$$P_1^{\text{Sch}}(z) := 1028584072611 + 127235412154z - 10561057362z^2 - 1864084076z^3 + 603469528z^4 - 60512739z^5. \quad (71)$$

$$P_2^{\text{Sch}}(z) := 1028055021271 + 127176631524z - 10562235361z^2 - 1857412419z^3 + 602438875z^4 - 60630782z^5. \quad (72)$$

$$P_3^{\text{Sch}}(z) := 1027949983376 + 127164960133z - 10562513303z^2 - 1856014917z^3 + 602162724z^4 - 60626649z^5. \quad (73)$$

$$P_4^{\text{Sch}}(z) := 1027929168654 + 127162644159z - 10562553224z^2 - 1855774812z^3 + 602146966z^4 - 60640921z^5. \quad (74)$$

$$P_5^{\text{Sch}}(z) := 1027925048882 + 127162186130z - 10562562991z^2 - 1855723560z^3 + 602140336z^4 - 60642455z^5. \quad (75)$$

Apply these schedules successively to the bounds

$$(K_0, \dots, K_6) := (2.29514500, 2.28844967, 2.28704152, 2.28675918, 2.28670312, 2.28669202, 2.28668982).$$

The j th schedule takes input K_j and has endpoint below K_{j+1} . Every displayed decimal is interpreted as an exact rational. For each input C , let $N(x), D(x)$ be the polynomials in Eqs. (53)–(54) and put $Y := N - kD$, with $m = m_j(s)$ and $k = k_j(s)$. Define the polynomials

$$\begin{aligned} R &:= mD - [m^2 - \tfrac{1}{2}s^2(1 - x^2)]Y, \\ F &:= mD^2 - [2m^2 - s^2(1 - x^2)]DY + m(m^2 - s^2)Y^2. \end{aligned}$$

Exact Bernstein subdivision certifies the same five domain polynomials, each with margin 10^{-6} , and establishes throughout the full square $0 \leq z, x \leq 1$ that

$$\begin{aligned} 2kD - N &\geq D/10, & Y &\geq D/10, & m - s &\geq 1/2, \\ R &\geq 10^{-8}, & F &\geq 10^{-8}. \end{aligned}$$

These are sufficient without approximating any square root. Indeed, with $y := Y/D > 0$ and $a = s^2(1 - x^2)/(2m)$,

$$\frac{R}{mD} = 1 - (m - a)y > 0,$$

and exact expansion gives

$$\frac{F}{mD^2} = [1 - (m - a)y]^2 - y^2(s^2x^2 + a^2) > 0.$$

Taking positive square roots yields $1 - yL(m, s, x) > 0$. Continuity on the compact square makes its minimum strictly positive, as required in Eq. (69).

The polynomials for the half-barrier and Y checks have bidegree $(5, 8)$, the $m - s$ check has bidegree $(7, 0)$, and the checks for R and F have bidegrees $(19, 10)$ and $(31, 16)$. The exact tensor Bernstein subdivision procedure described above verifies all six schedules; its greatest subdivision depth is 14. Direct expansion also gives the polynomial identity

$$R^2 - Y^2[m^2s^2x^2 + \tfrac{1}{4}s^4(1 - x^2)^2] = mF,$$

the antiderivative coefficients, $m_0 > Cs_0$, and every strict rational endpoint inequality. The new checks process 2218470 Bernstein coefficients. Starting with the admissible constant 2.295145, six applications of Lemma 4.17 give

$$m_5(1) = \frac{1920819441860480751179}{8400000000000000000000} < 2.28668982 < \frac{228669}{100000} = 2.28669.$$

This proves the intermediate admissible bound 2.28669. The certificate proves the scalar inequalities on their full domains; the preceding compression and finite-copy lemmas supply the matrix argument.

4.6 The positive sixth-moment refinement

A positive sixth-moment remainder strengthens the moment relation and gives a smaller initial barrier for the next schedules.

Lemma 4.18 (Positive sixth-moment remainder). *Write $D_i = w_i w_i^*$ and suppose $\sum_{i=1}^{N_D} D_i^2 = qI$. Define W, X, Y, Ψ by the same formulas as in the proof of Lemma 4.10, and put $\mu_{2j} = \text{tr}[PT^{2j}]/2$. Then*

$$\mathcal{E} := qW + \Psi(W) - X - Y \succeq 0. \quad (76)$$

Consequently

$$\mu_6 \leq 4q\mu_4 - 3q^3. \quad (77)$$

Proof. Let $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \overline{\mathbb{C}^d}$ and define

$$Lx = \sum_{j=1}^{N_D} (w_j^* x) w_j \otimes w_j \otimes \overline{w_j}.$$

Direct tensor inner products show that $L^*L = W$. On $\mathcal{H}$ put

$$C_i = D_i \otimes I \otimes I, \quad E_i = I \otimes I \otimes \overline{D_i}.$$

Then $\sum_{i=1}^{N_D} C_i^2 = \sum_{i=1}^{N_D} E_i^2 = qI$. Two further contractions give

$$\sum_{i=1}^{N_D} L^* C_i L D_i = Y, \quad \sum_{i=1}^{N_D} L^* E_i L D_i = X. \quad (78)$$

For clarity, writing $G_{ij} = w_i^* w_j$, the coefficient of $w_a w_c^*$ in the first expression is

$$\sum_{b=1}^{N_D} |G_{ab}|^2 |G_{bc}|^2 G_{ac},$$

which is the coefficient in Y . In the second it is

$$\sum_{b=1}^{N_D} G_{ab}^2 G_{bc}^2 G_{ca},$$

which is the coefficient in X . Thus (78) is valid over the complex field, with the stated conjugations.

Expanding squares now gives

$$\sum_{i=1}^{N_D} (C_i L - L D_i)^* (C_i L - L D_i) = qW + \Psi(W) - 2Y, \quad (79)$$

$$\sum_{i=1}^{N_D} (E_i L - L D_i)^* (E_i L - L D_i) = qW + \Psi(W) - 2X. \quad (80)$$

Average these identities to obtain (76). Rescale Eq. (42) from $q = 1$ and substitute to obtain the exact remainder formula

$$\mu_6 = v^* (5q^3 I - 4qW - \mathcal{E}) v. \quad (81)$$

The fourth-moment identity $\mu_4 = 2q^2 - v^* W v$ then proves Eq. (77). $\square$

The positive remainder gives a sharper cubic initial barrier. Normalize a full family $B = (B_i)$ by $\sum_{i=1}^n B_i^2 = I$. For a unit vector v , define the full-family parameter

$$\zeta_v = \left(\sum_{i=1}^n (v^* B_i v)^2 \right)^{1/2} \in [0, 1].$$

Let $C > 2$ be admissible, and choose $0 < s_0 < 1$, $m_0 > C s_0$. Set

$$q = \frac{s_0^2}{m_0^2}, \quad c = C^2, \quad r = cq, \quad b_z = q \frac{2c - 5 + (4 - c)z^4}{c - 2 + z^4}. \quad (82)$$

The symbol b_z here is a scalar interpolation node, not the trace of a matrix direction. Require, for all $z \in [0, 1]$,

$$0 < b_z < r < 1, \quad 1 - r - 2b_z + 4q \geq 0. \quad (83)$$

Define

$$U_z^{(6)}(C; m_0, s_0) = \frac{2}{m_0} \left(1 + \frac{q(1 - b_z - r) + (2 - z^4)q^2}{(1 - b_z)(1 - r)} \right). \quad (84)$$

Lemma 4.19 (Sixth-moment initial barrier). *Under these conditions, $(m_0 I, 1)$ lies in the cone of $H_{s_0 B}$, and*

$$\mathcal{D}_{\beta v v^*} \log H_{s_0 B}(m_0 I, 1) \leq \beta U_{\zeta_v}^{(6)}(C; m_0, s_0).$$

Moreover $U_z^{(6)}$ is nonincreasing in $z \in [0, 1]$.

Proof. Cone membership follows from admissibility and $m_0 > C s_0$. Apply Lemmas 4.18 and 4.12 to $D_i = s_0 B_i / m_0$. The spectral measure of T^2 weighted by $P/2$ is a probability measure on $[0, r]$: admissibility and evenness give $\|T\| \leq C\sqrt{q}$. For its scalar variable X ,

$$\mathbb{E}[X] = q, \quad \mathbb{E}[X^2] \leq (2 - z^4)q^2, \quad \mathbb{E}[X^3] \leq 4q \mathbb{E}[X^2] - 3q^3, \quad z = \zeta_v. \quad (85)$$

For $0 < b < r < 1$, put $D_0 = (1 - r)(1 - b)^2$ and

$$Q_b(x) = 1 + \left(1 + \frac{rb^2}{D_0}\right)x + \frac{1 - r - 2b}{D_0}x^2 + \frac{x^3}{D_0}.$$

The exact identity

$$Q_b(x) - \frac{1}{1 - x} = \frac{x(r - x)(x - b)^2}{D_0(1 - x)} \geq 0 \quad (0 \leq x \leq r) \quad (86)$$

proves majorization. The coefficient of x^2 can be negative, so it would be invalid to substitute the second and third moment bounds independently. Instead first use the coupled inequality in (85). The coefficient multiplying $\mathbb{E}[X^2]$ then becomes

$$\frac{1 - r - 2b + 4q}{D_0} \geq 0.$$

Only now substitute the upper bound on $\mathbb{E}[X^2]$. This gives

$$\mathbb{E}\left[\frac{1}{1 - X}\right] \leq 1 + \left(1 + \frac{rb^2}{D_0}\right)q + \frac{1 - r - 2b}{D_0}(2 - z^4)q^2 + \frac{(5 - 4z^4)q^3}{D_0}.$$

Taking $b = b_z$ reduces the right side to the bracket in (84). Odd directional moments vanish, so the original barrier is $2\beta/m_0$ times this expectation. Here β is the rank-one direction scale, denoted b in the other compression lemmas; the scalar node in this majorant is b_z .

For monotonicity, write $w = z^4$ and denote the last displayed bound by $F(b, w)$. Direct differentiation gives $\partial_b F(b_z, w) = 0$. Hence

$$\frac{d}{dw} F(b_z, w) = -q^2 \frac{1 - r - 2b_z + 4q}{(1 - r)(1 - b_z)^2} \leq 0.$$

This proves the final assertion. $\square$

We recall the compression estimate with its hypotheses, in particular the cone-membership condition needed near the endpoint.

Lemma 4.20 (Variance-retaining compression). *Suppose C is admissible, $V_D \preceq \theta I$, $m > 0$, and $(mI, 1)$ is in the cone of H_D . For a direction $B = \beta vv^*$, set*

$$q_v = v^* V_D v, \quad \nu^2 = \sum_{i=1}^{N_D} (v^* D_i v)^2, \quad e = \frac{q_v - \nu^2}{m + C\sqrt{\theta}}.$$

The smaller of the two directional eigenvalues satisfies

$$\ell_B \geq \frac{\beta}{m - e + \sqrt{\nu^2 + e^2}}.$$

No assumption $m > C\sqrt{\theta}$ is needed.

Proof. Use the ternary pencil from Lemma 4.16. In this pencil let $A_0 = PTP|_{\text{ran } P}$ and $M_2 = PT^2P|_{\text{ran } P}$. The coefficient identities give

$$\text{tr}[A_0] = 0, \quad A_0^2 = \nu^2 I_2, \quad M_2 = q_v I_2 + R_0, \quad \text{tr}[R_0] = 0.$$

Both the trace and determinant of $P(I + tT)^{-1}P|_{\text{ran } P}$ are even in t . To see the coefficient implication, expand the compression as $I - tA_0 + t^2M_2 - t^3M_3 + O(t^4)$. Evenness of its trace gives $\text{tr}[M_3] = 0$, and the cubic coefficient of its determinant is then $\text{tr}[A_0M_2]$; this must vanish. Thus $\text{tr}[A_0M_2] = 0$. For traceless symmetric 2×2 matrices this means $A_0R_0 + R_0A_0 = 0$. Also

$$M_2 - A_0^2 = PT(I - P)TP \succeq 0, \quad \|R_0\| \leq q_v - \nu^2.$$

Admissibility gives $\|T\| \leq C\sqrt{\theta}$, while actual cone membership gives $mI + T \succ 0$. Write $Z = PT(I - P)$ and $T_Q = (I - P)T(I - P)$ on the complement. The inverse of the compression of $(mI + T)^{-1}$ is

$$S = mI_2 + A_0 - Z(mI + T_Q)^{-1}Z^*.$$

Since $0 \prec mI + T_Q \preceq (m + C\sqrt{\theta})I$,

$$0 \prec S \preceq (m - e)I_2 + A_0 - \frac{R_0}{m + C\sqrt{\theta}}.$$

Anticommutation and the bound on R_0 give

$$\lambda_{\max}(S) \leq m - e + \sqrt{\nu^2 + e^2}.$$

The directional eigenvalues are the eigenvalues of βS^{-1} , proving the assertion. This argument uses actual cone membership, not a new root bound at m . $\square$

For a complete isotropic scaled family, define

$$e(C; m, s, z) = \frac{s^2(1 - z^2)}{m + Cs}, \tag{87}$$

$$\mathcal{L}(C; m, s, z) = \frac{1}{m - e(C; m, s, z) + \sqrt{s^2 z^2 + e(C; m, s, z)^2}}. \tag{88}$$

The denominator is at least $m > 0$.

Theorem 4.21 (Sixth-moment schedule). *Let C, m_0, s_0 satisfy the assumptions of Lemma 4.19. Let k be continuous on $[s_0, 1]$, and set*

$$m(s) = m_0 + \int_{s_0}^s 2uk(u) du.$$

Suppose, for some $\gamma > 0$, that all $s \in [s_0, 1]$ and $z \in [0, 1]$ satisfy

$$k(s) \geq U_0^{(6)}(C; m_0, s_0)/2, \quad (89)$$

$$k(s) - U_z^{(6)}(C; m_0, s_0) + \mathcal{L}(C; m(s), s, z) \geq \gamma. \quad (90)$$

Then $m(1)$ is admissible.

Proof. The polynomial-level padding proved above reduces the claim to $\sum_{i=1}^n B_i^2 = I$; discard zero matrices. Write $B_i = \beta_i v_i v_i^*$, with $\|v_i\| = 1$, and set $\zeta_i^2 = \sum_{j=1}^n (v_i^* B_j v_i)^2$.

For an integer N , put

$$\Delta = \frac{1 - s_0^2}{N}, \quad t_r = s_0^2 + r\Delta, \quad s_r = \sqrt{t_r}, \quad k_r = k(s_r), \quad m_r = m_0 + \Delta \sum_{j=1}^r k_j.$$

Start with $H_{s_0 B}$ and append N remaining copies $\sqrt{\Delta} B_i$ in each direction. Give copy (r, i) initial coordinate $-\delta_{r,i}$, where $\delta_{r,i} = k_r \beta_i \sqrt{\Delta}$, and fix the spectral coordinate at $x = m_N$. Since $\sum_{i=1}^n \beta_i B_i = I$, the initial matrix argument is precisely $m_0 I$. Initial cone membership and the unused barriers are supplied by Lemma 4.19; an unused copy of direction i has barrier at most

$$\beta_i \sqrt{\Delta} U_{\zeta_i}^{(6)}(C; m_0, s_0).$$

Before an update in round r , let D denote all processed directions, including the initial family. The current matrix argument M satisfies $M \preceq m_r I$, while

$$(t_r - \Delta)I \preceq V_D \preceq t_r I, \quad \sum_{j=1}^{N_D} (v_i^* D_j v_i)^2 \leq t_r \zeta_i^2. \quad (91)$$

Cone comparison bounds the smaller directional eigenvalue at M below by the one at $m_r I$. Apply Lemma 4.20 there. Its e is at least

$$e_{r,N}(z) = \frac{\max\{0, t_r(1 - z^2) - \Delta\}}{m_r + C s_r},$$

and its ν is at most $s_r \zeta_i$. Since $m - e + \sqrt{\nu^2 + e^2}$ decreases in $e \geq 0$ and increases in $\nu \geq 0$, the smaller eigenvalue of the copy is at least

$$\frac{\beta_i \sqrt{\Delta}}{m_r - e_{r,N}(\zeta_i) + \sqrt{s_r^2 \zeta_i^2 + e_{r,N}(\zeta_i)^2}}. \quad (92)$$

This explicitly includes the incomplete-round error.

Uniform Riemann-sum convergence gives $\max_r |m_r - m(s_r)| \rightarrow 0$. Also

$$0 \leq e(C; m_r, s_r, z) - e_{r,N}(z) \leq \frac{\Delta}{m_r + C s_r}.$$

All $m_r \geq m_0 > 0$. Thus the normalized lower bound in (92) converges uniformly to (88). The strict slack in (90) ensures, for sufficiently large finite N , that the maintained unused barrier u and smaller eigenvalue ℓ satisfy $u - \ell \leq \delta_{r,i}$. Monotonicity $U_z^{(6)} \leq U_0^{(6)}$ and (89) give $u \leq 2\delta_{r,i}$.

These are exactly the two hypotheses of Lemma 2.5. It preserves above-root membership and every remaining unused barrier. This proves the induction without presupposing cone membership at later rounds.

At the end, the operators in each original direction combine according to

$$(1 - \frac{s_0^2}{2}\mathcal{D}_{B_i}^2)(1 - \frac{\Delta}{2}\mathcal{D}_{B_i}^2)^N \det(X)^2 = (1 - \frac{1}{2}\mathcal{D}_{B_i}^2) \det(X)^2.$$

Indeed $\mathcal{D}_{B_i}^4 \det(X)^2 = 0$, all the directional operators commute, and $s_0^2 + N\Delta = 1$. The identity remains valid after operators in other directions. Therefore the original polynomial has largest root at most m_N . Taking $N \rightarrow \infty$ gives $m(1)$ for every fixed family. No selected signing of copies is used or transferred. $\square$

The preceding argument proves that 2.28669 is admissible, hence so is 2.2904. Starting from this deliberately weaker admissible input, use the following four stages. All terminating decimals denote exact rational numbers. Define the successive admissible bounds by

$$(\widehat{C}_0, \dots, \widehat{C}_4) := (2.2904, 2.2832, 2.2818, 2.2814, 2.28128).$$

For stages $j = 1, \dots, 4$, use $(m_{0,j}, s_{0,j})$ in the order

$$(1.7336, 0.7038), \quad (1.7391, 0.7084), \quad (1.7411, 0.7098), \quad (1.7414, 0.7100).$$

Stage j takes input $\widehat{C}_{j-1}$ and has endpoint below $\widehat{C}_j$. Let $x = (s - s_0)/(1 - s_0)$ and $L = 1 - s_0$. Define

$$k(s) = 10^{-12} \sum_{i=0}^9 a_i x^i,$$

$$m(s) = m_0 + 2L10^{-12} \sum_{i=0}^9 a_i \left(\frac{s_0 x^{i+1}}{i+1} + \frac{Lx^{i+2}}{i+2} \right).$$

In stage j , the coefficients are $a_i := [x^i]P_j^+(x)$, where

$$\begin{aligned} P_1^+(x) := & 1025092053946 + 128487087852x - 10542200968x^2 \\ & - 2018813096x^3 + 780760118x^4 - 153635330x^5 \\ & + 30254428x^6 - 5923332x^7 + 747672x^8 \\ & - 31485x^9. \end{aligned} \tag{93}$$

$$\begin{aligned} P_2^+(x) := & 1026405946993 + 126332907265x - 10222524143x^2 \\ & - 1917218101x^3 + 726007282x^4 - 138389131x^5 \\ & + 26459611x^6 - 5114284x^7 + 638890x^8 \\ & - 26049x^9. \end{aligned} \tag{94}$$

$$\begin{aligned}
P_3^+(x) := & 1026508448809 + 125614921294x - 10114462286x^2 \\
& - 1886824377x^3 + 709484719x^4 - 133863293x^5 \\
& + 25343292x^6 - 4876381x^7 + 606932x^8 \\
& - 24464x^9.
\end{aligned} \tag{95}$$

$$\begin{aligned}
P_4^+(x) := & 1026418425685 + 125498403161x - 10099117644x^2 \\
& - 1881213392x^3 + 706904260x^4 - 133266353x^5 \\
& + 25211714x^6 - 4847437x^7 + 602622x^8 \\
& - 24198x^9.
\end{aligned} \tag{96}$$

With q, c, r as in Eq. (82), put

$$\begin{aligned}
d(z) &= c - 2 + z^4, & n(z) &= q[2c - 5 + (4 - c)z^4], \\
\mathcal{N}(z) &= 2[(1 - r)(d - n) + q((1 - r)d - n) + (2 - z^4)q^2d], \\
\mathcal{D}(z) &= m_0(1 - r)(d - n).
\end{aligned}$$

Thus $b_z = n/d$ and $U_z^{(6)} = \mathcal{N}/\mathcal{D}$. Exact Bernstein coefficients establish throughout $0 \leq z \leq 1$ that

$$d > 0, \quad n > 0, \quad rd - n > 0, \quad d - n > 0, \quad (1 - r + 4q)d - 2n > 0.$$

These imply the node and coefficient conditions in Eq. (83). Set

$$\begin{aligned}
F &= \mathcal{N} - k\mathcal{D}, & E &= \mathcal{D} - mF, & h &= m + Cs, \\
\mathcal{P} &= h(E^2 - s^2z^2F^2) + 2Es^2(1 - z^2)F.
\end{aligned}$$

For each stage, exact Bernstein subdivision establishes on the entire square $0 \leq x, z \leq 1$ that

$$k - U_0^{(6)}/2 > 1/10, \quad F > 1/10, \quad E > 1/100, \quad \mathcal{P} > 10^{-7}. \tag{97}$$

To deduce the radical inequality, use the exact polynomial identity

$$h\mathcal{P} = [hE + s^2(1 - z^2)F]^2 - F^2[s^2z^2h^2 + s^4(1 - z^2)^2].$$

The first bracket and F are positive. Taking positive square roots gives, with $e = s^2(1 - z^2)/h$,

$$E + eF > F\sqrt{s^2z^2 + e^2}, \quad \mathcal{D} > F[m - e + \sqrt{s^2z^2 + e^2}].$$

Since the bracket and $\mathcal{D}$ are positive, this is $k - U_z^{(6)} + \mathcal{L}(C; m, s, z) > 0$. Continuity on the compact square supplies the uniform positive γ in Theorem 4.21.

For completeness, these are certificates over full rectangles. If $p(x, z) = \sum_{i=0}^n \sum_{j=0}^m p_{ij}x^i z^j$ has bidegree (n, m) , its tensor Bernstein coefficients are

$$\beta_{ab} = \sum_{i=0}^a \sum_{j=0}^b p_{ij} \frac{\binom{a}{i}}{\binom{n}{i}} \frac{\binom{b}{j}}{\binom{m}{j}}.$$

The basis is nonnegative and sums to one. The five domain polynomials have bidegree $(0, 4)$; the first three margin polynomials in Eq. (97) have bidegrees $(9, 0)$, $(9, 8)$, and $(20, 8)$. Their Bernstein coefficients are positive on the whole square. For $\mathcal{P} - 10^{-7}$, of bidegree $(51, 18)$, seven exact midpoint

subdivisions in z give positive coefficients on all 128 strips $[0, 1] \times [j/128, (j+1)/128]$. Each stage thus checks

$$5 \cdot 5 + 10 + 90 + 189 + 128 \cdot 52 \cdot 19 = 126778$$

coefficients. The four stages require 507112 coefficient checks. Exact differentiation, expansion, and rational substitution verify the antiderivative, radical-clearing identity, stationarity and simplification of the cubic barrier, $m_0 > Cs_0$, and every strict rational endpoint comparison. The exact endpoints are

$$\begin{aligned} m_1(1) &= \frac{65924385244054794774599}{2887500000000000000000000} < 2.2832 \\ m_2(1) &= \frac{21960841725386168489499}{9625000000000000000000000} < 2.2818 \\ m_3(1) &= \frac{71862813588176222500673}{31500000000000000000000000} < 2.2814 \\ m_4(1) &= \frac{45169280617678560841}{19800000000000000000000000} < 2.28128. \end{aligned}$$

In particular,

$$\frac{7129}{3125} - m_4(1) = \frac{63382321439159}{19800000000000000000000000} > 0.$$

Four applications of Theorem 4.21 prove that $7129/3125 = 2.28128$ is admissible. This is the input to the eighth-moment schedules below, whose output in turn initializes the tenth-moment proof of Theorem 4.1.

4.7 The eighth-moment refinement

We derive a coupled eighth-moment inequality, use it in a quartic barrier estimate, and certify the resulting schedules.

First normalize $\sum_{i=1}^{N_D} D_i^2 = I$, and write $D_i = w_i w_i^*$. Define

$$\begin{aligned} \Psi(X) &= \sum_{i=1}^{N_D} D_i X D_i, & W &= \sum_{i,j=1}^{N_D} \text{tr}[D_i D_j] D_i D_j, \\ X &= \sum_{i,j,k=1}^{N_D} \text{tr}[D_i D_j D_k] D_i D_j D_k, & Y &= \sum_{i=1}^{N_D} \Psi(D_i)^2. \end{aligned}$$

Rank one also gives

$$Y = \sum_{i,j,k=1}^{N_D} \overline{\text{tr}[D_i D_j D_k]} D_i D_j D_k.$$

For a word $a_1 \cdots a_{2k}$ in which each label in $\{1, \dots, k\}$ occurs twice, write

$$\mathsf{T}_{a_1 \cdots a_{2k}} := \sum_{i_1, \dots, i_k=1}^{N_D} D_{i_{a_1}} \cdots D_{i_{a_{2k}}}.$$

The indices are summed independently, with no distinctness condition. In particular,

$$\mathsf{T}_{11} = I, \quad \mathsf{T}_{1212} = W, \quad \mathsf{T}_{123231} = \Psi(W), \quad \mathsf{T}_{123123} = X, \quad \mathsf{T}_{121323} = Y.$$

These identities preserve the order of all complex Gram factors.

For a unit vector v , a ternary determinantal representation takes the form

$$H_D(xI + hvv^*, t) = \det(xI_{2d} + hP + tT), \quad (98)$$

where P is a rank-two orthogonal projection and T is real symmetric. The representation can depend on v . Only this three-variable restriction is represented.

Lemma 4.22. *There are Hermitian matrix expressions $\mathcal{M}_{2k}$, independent of the choice of v , such that*

$$\frac{1}{2} \operatorname{tr}[PT^{2k}] = v^* \mathcal{M}_{2k} v.$$

All odd directional moments vanish. The first three even expressions are

$$\mathcal{M}_2 = I, \quad (99)$$

$$\mathcal{M}_4 = 2I - W, \quad (100)$$

$$\mathcal{M}_6 = 5I - 5W - \Psi(W) + X + Y. \quad (101)$$

The eighth expression is

$$\begin{aligned} \mathcal{M}_8 = & 14\mathsf{T}_{11} - \mathsf{T}_{12312434} - 20\mathsf{T}_{1212} \\ & - \mathsf{T}_{12314234} + 7\mathsf{T}_{121323} - \mathsf{T}_{12314243} \\ & + 7\mathsf{T}_{123123} + \mathsf{T}_{12314324} - 7\mathsf{T}_{123231} \\ & + \mathsf{T}_{12324341} + \mathsf{T}_{12123434} - \mathsf{T}_{12341234} \\ & + \mathsf{T}_{12132434} + \mathsf{T}_{12342341} - \mathsf{T}_{12134234} \\ & + \mathsf{T}_{12343412} - 2\mathsf{T}_{12134243} - \mathsf{T}_{12343421} \\ & + 2\mathsf{T}_{12134342}. \end{aligned} \quad (102)$$

Proof. Differentiation of (98) gives

$$\mathcal{D}_{vv^*} \log H_D(I, t) = \operatorname{tr}[P(I + tT)^{-1}].$$

The left side is even in t , proving the odd-moment statement. For the coefficient calculation let $G_X(i, j) = w_i^* X^{-1} w_j$. The rank-one determinant expansion gives

$$\frac{H_D(X, t)}{\det(X)^2} = \sum_{S \subseteq [N_D]} (-t^2)^{|S|} \det(G_X[S])^2. \quad (103)$$

For degree $2k$, expand the square of a determinant using two permutations $\pi, \rho \in S_k$, allow repeated indices, and divide the ordered sum by $k!$. The contribution is

$$\frac{(-t^2)^k}{k!} \sum_{i_1, \dots, i_k=1}^{N_D} \sum_{\pi, \rho \in S_k} \operatorname{sgn}(\pi) \operatorname{sgn}(\rho) \prod_{a=1}^k G_X(i_a, i_{\pi(a)}) G_X(i_a, i_{\rho(a)}).$$

Repeated indices give zero after summing the determinant expansion. In the formal logarithm only pairs whose joint action is transitive remain: each pair splits uniquely into its joint orbits, and the index sums factor over those orbits. This is the usual exponential-formula identity, here used only through $k = 4$.

Represent a Gram entry by a directed edge. Differentiating an edge $a \rightarrow b$ in the direction vv^* replaces it by

$$-(w_{ia}^* v)(v^* w_{ib}),$$

or, at the matrix level, by $-w_{ib} w_{ia}^*$. Mark the removed edge. At $X = I$, eliminate a self-loop vertex using

$$\sum_{i=1}^{N_D} \|w_i\|^2 w_i w_i^* = I,$$

joining its adjacent edges in their original directions. This rule also applies when one adjacent edge is the marked edge. Relabel dummy indices to compare the remaining directed graphs. Rank one identifies every resulting graph with a word in Eq. (102) or a lower-order word above.

The signed marked-edge sum is multiplied by $(-1)^{k+1}/(2k!)$. For $k = 1, 2, 3, 4$, the numbers of transitive permutation pairs are 1, 3, 26, 426, respectively. Marking one of the $2k$ edges gives 2, 12, 156, 3408 terms, respectively. Collecting the terms gives the displayed identities and Eq. (102). These finite rules compare the full directed-graph coefficient dictionaries. In particular no commutativity or real-vector restriction is used. $\square$

The eighth directional moment satisfies the following coupled inequality.

Theorem 4.23. For $\sum_{i=1}^{N_D} D_i^2 = qI$, with $q > 0$, the pencil (98) satisfies

$$\frac{1}{2} \text{tr}[PT^8] - 4q \frac{1}{2} \text{tr}[PT^6] + \frac{13}{4} q^2 \frac{1}{2} \text{tr}[PT^4] \leq \frac{1}{2} q^4. \quad (104)$$

Proof. It suffices to prove the result for $q = 1$ and then rescale D_i by $q^{-1/2}$. First,

$$Y - W^2 \succeq 0. \quad (105)$$

Indeed define maps $Vx = (D_i x)_i$ and $Sx = (\Psi(D_i)x)_i$. Isotropy gives $V^*V = I$, while

$$V^*S = \sum_{i=1}^{N_D} D_i \Psi(D_i) = W, \quad S^*S = \sum_{i=1}^{N_D} \Psi(D_i)^2 = Y.$$

The matrix W is Hermitian, so $Y - W^2 = S^*(I - VV^*)S \succeq 0$.

Let

$$\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \overline{\mathbb{C}^d}, \quad Lx = \sum_{j=1}^{N_D} (w_j^* x) w_j \otimes w_j \otimes \overline{w_j}.$$

Direct inner products give $L^*L = W$. Define the three coordinate actions

$$C_{1,i} = D_i \otimes I \otimes I, \quad C_{2,i} = I \otimes D_i \otimes I, \quad C_{3,i} = I \otimes I \otimes \overline{D_i}.$$

For each pair (i, j) , construct twenty maps from $\mathbb{C}^d$ into $\mathcal{H}$ as follows. For $a = 3, 2, 1$, put $h_a := (3-a)(4-a) + 1$. For $r = 0, \dots, 3-a$, define

$$Z_{h_a+r}(i, j) := C_{a,i} C_{3-r,j} L, \quad (106)$$

$$Z_{h_a+4-a+r}(i, j) := C_{a,j} C_{3-r,i} L. \quad (107)$$

These are maps $Z_1, \dots, Z_{12}$. Complete the family by

$$Z_{13+r}(i, j) := C_{3-r,j} L D_i \quad (r = 0, 1, 2), \quad (108)$$

$$Z_{16}(i, j) := L D_i D_j, \quad (109)$$

$$Z_{a+4}(i, j) := Z_a(j, i) \quad (a = 13, \dots, 16). \quad (110)$$

We construct the Gram coefficients from a polynomial. For each ordered pair (a, b) , let Γ_{ab} be the reduced directed graph of $\sum_{i,j} Z_a(i, j)^* Z_b(i, j)$, using the contraction rules in Lemma 4.22. Ordinary tensor factors contribute edges from the left output to the right output, and the conjugate third factor reverses this direction. The input factors determine the marked edge. Sum the four indices independently, eliminate self-loops by isotropy, and identify graphs under relabeling of their dummy indices. For a formal word expression F , write $[\Gamma]F$ for its coefficient after these reductions.

Define the formal polynomial

$$\begin{aligned} 100h = & 8994\mathsf{T}_{1212} + 4189\mathsf{T}_{123231} + 3805\mathsf{T}_{121323} + 2841\mathsf{T}_{123123} \\ & + 4382\mathsf{T}_{12343421} + 450\mathsf{T}_{12324341} + 895\mathsf{T}_{12132434} - 65\mathsf{T}_{12134243} \\ & - 118(\mathsf{T}_{12134234} + \mathsf{T}_{12312434}) + 4175\mathsf{T}_{12343412} - 910\mathsf{T}_{12342341} \\ & + 6584\mathsf{T}_{12134342} - 326\mathsf{T}_{12341234} - 862\mathsf{T}_{12314234} + 436\mathsf{T}_{12314324} - 670\mathsf{T}_{12314243}. \end{aligned}$$

Set $H_{ab} := [\Gamma_{ab}]h$. This defines a symmetric rational 20×20 matrix. Its characteristic polynomial determines

$$p(x) := \det(xI_{20} - H), \quad f(x) := \frac{p(0) - p(x)}{xp(0)}, \quad R := f(H).$$

The exact pivot calculation below gives $H \succ 0$, so $p(0) > 0$. Thus f is a polynomial of degree 19, and Cayley–Hamilton gives $HR = I_{20}$.

Put

$$F := \frac{1}{2}I - \mathcal{M}_8 + 4\mathcal{M}_6 - \frac{13}{4}\mathcal{M}_4 - Y + W^2, \quad m_\Gamma := |\{(c, d) : \Gamma_{cd} = \Gamma\}|.$$

Define every Gram coefficient by

$$Q_{ab} := R_{ab} + \frac{[\Gamma_{ab}]F - \sum_{\Gamma_{cd}=\Gamma_{ab}} R_{cd}}{m_{\Gamma_{ab}}}. \quad (111)$$

The 400 ordered pairs give 17 directed graph classes, which also contain the support of F . Reversing all edges interchanges Γ_{ab} and Γ_{ba} ; the coefficients of h and F agree on reversed classes. Consequently Q is symmetric. Summing (111) over a class gives $\sum_{\Gamma_{ab}=\Gamma} Q_{ab} = [\Gamma]F$, proving the exact identity

$$\frac{1}{2}I - \mathcal{M}_8 + 4\mathcal{M}_6 - \frac{13}{4}\mathcal{M}_4 = Y - W^2 + \sum_{i,j=1}^{N_D} \sum_{a,b=1}^{20} Q_{ab} Z_a(i, j)^* Z_b(i, j). \quad (112)$$

For positivity, apply the following rational recurrences to $A = H$ and $A = Q$, in the displayed order of the twenty maps:

$$d_j = A_{jj} - \sum_{k=1}^{j-1} \ell_{jk}^2 d_k, \quad \ell_{ij} = \frac{A_{ij} - \sum_{k=1}^{j-1} \ell_{ik} d_k \ell_{jk}}{d_j} \quad (i > j), \quad \ell_{jj} = 1.$$

Exact arithmetic gives $d_j > 2$ for H and $d_j > 1/250$ for Q , for every $j = 1, \dots, 20$. These finite checks use only the displayed coefficients of h , the graph reductions, and the recurrences:

each comparison is an integer comparison after clearing positive denominators. They give $A = L \operatorname{diag}(d_1, \dots, d_{20}) L^\top$ with $L = (\ell_{ij})$ unit lower triangular, and hence $H, Q \succ 0$. Together with (105), the Gram identity proves (104) for $q = 1$ after taking $v^*(\cdot)v$. Homogeneity gives the stated factors of q . $\square$

Lemma 4.24 (Directional fourth-moment defect). *For a unit vector v , put $\nu_v^2 = \sum_{i=1}^{N_D} (v^* D_i v)^2$. Then*

$$v^* W v \geq \nu_v^4.$$

Proof. With the same map L , the tensor $p = v \otimes v \otimes \bar{v}$ is a unit vector and $\langle p, Lv \rangle = \nu_v^2$. Cauchy–Schwarz and $L^* L = W$ prove the claim. This is the estimate of Lemma 4.12. $\square$

A quartic majorant converts this coupled inequality into an initial barrier. Normalize the full family by $\sum_{i=1}^n B_i^2 = I$ and fix the directional parameter

$$z = \zeta_v := \left(\sum_{i=1}^n (v^* B_i v)^2 \right)^{1/2} \in [0, 1].$$

Let $C > 2$ be admissible, choose $0 < s_0 < 1$ and $m_0 > C s_0$, and set

$$q = \frac{s_0^2}{m_0^2}, \quad r = C^2 q < 1, \quad \alpha = \frac{1-r+4q}{2}, \quad d = 1-\alpha, \quad w = z^4. \quad (113)$$

For the normalized family $D_i = s_0 B_i / m_0$, let X be the scalar variable of the spectral measure of T^2 weighted by $P/2$. Admissibility gives its support in $[0, r]$. The exact moment formulas, Lemma 4.24, and Theorem 4.23 give

$$\mathbb{E}[X] = q, \quad \mathbb{E}[X^2] \leq (2-w)q^2, \quad \mathbb{E}[X^4] - 4q\mathbb{E}[X^3] + \frac{13}{4}q^2\mathbb{E}[X^2] \leq \frac{1}{2}q^4. \quad (114)$$

The last condition is used as a coupled inequality.

For a real parameter β with $d + \beta \neq 0$, put

$$p_\beta(x) = x^2 - \alpha x + \beta, \quad \Delta = (1-r)(d+\beta)^2 > 0.$$

The expression

$$Q_\beta(x) = \frac{1 + (r-x)p_\beta(x)^2/\Delta}{1-x}$$

is a polynomial of degree four: its numerator vanishes at $x = 1$. Moreover

$$Q_\beta(x) - \frac{1}{1-x} = \frac{(r-x)p_\beta(x)^2}{\Delta(1-x)} \geq 0 \quad (0 \leq x \leq r). \quad (115)$$

Writing $Q_\beta = \sum_{j=0}^4 c_j x^j$, the coefficients are

$$c_4 = \Delta^{-1}, \quad c_3 = (1-r-2\alpha)\Delta^{-1} = -4q\Delta^{-1}, \quad (116)$$

$$c_2 = ((1-r)(1-2\alpha) + \alpha^2 + 2\beta)\Delta^{-1}, \quad (117)$$

$$c_1 = ((1-r)(1-2\alpha + \alpha^2 + 2\beta) - 2\alpha\beta)\Delta^{-1}, \quad (118)$$

$$c_0 = 1 + r\beta^2\Delta^{-1}. \quad (119)$$

In particular, c_3 is negative; independently substituting upper bounds for all moments would be invalid.

First substitute the coupled last inequality in (114). The coefficient of $\mathbb{E}[X^3]$ is exactly zero. If

$$(1-r)(1-2\alpha) + \alpha^2 + 2\beta - \frac{13}{4}q^2 \geq 0, \quad (120)$$

we can then substitute the bound on $\mathbb{E}[X^2]$. The resulting expectation bound is

$$\mathbb{E}[Q_\beta(X)] \leq \frac{\beta^2 + n_1\beta + n_0}{(1-r)(d+\beta)^2}, \quad (121)$$

where

$$n_1(w) = 2((1-r)d + q(1-r-\alpha) + (2-w)q^2), \quad (122)$$

$$\begin{aligned} n_0(w) &= (1-r)d^2(1+q) \\ &\quad + (2-w)q^2((1-r)(1-2\alpha) + \alpha^2) + q^4(-6 + \frac{13}{4}w). \end{aligned} \quad (123)$$

Both are affine in w . Define

$$B(w) = 2d - n_1(w), \quad A(w) = 2n_0(w) - dn_1(w), \quad (124)$$

$$J(w) = d^2 - dn_1(w) + n_0(w), \quad \beta(w) = A(w)/B(w). \quad (125)$$

The functions $A(w)$, $B(w)$, and $J(w)$ are scalars.

At this node the right side of (121) simplifies to

$$\frac{4n_0 - n_1^2}{4(1-r)J}.$$

Indeed $d + \beta = 2J/B$, and direct substitution proves the identity. Thus define

$$U_z^{(8)}(C; m_0, s_0) = \frac{4n_0(z^4) - n_1(z^4)^2}{2m_0(1-r)J(z^4)}. \quad (126)$$

Lemma 4.25. *Suppose, for all $0 \leq w \leq 1$, that $B(w) > 0$, $J(w) > 0$, and*

$$((1-r)(1-2\alpha) + \alpha^2 - \frac{13}{4}q^2)B(w) + 2A(w) \geq 0.$$

Then $(m_0I, 1)$ belongs to the cone of H_{s_0B} , and every rank-one direction bvv^* satisfies

$$\mathcal{D}_{bvv^*} \log H_{s_0B}(m_0I, 1) \leq bU_{\zeta_v}^{(8)}(C; m_0, s_0).$$

The function $U_z^{(8)}$ is nonincreasing in z .

Proof. Cone membership follows from admissibility and $m_0 > Cs_0$. The assumptions imply $d + \beta = 2J/B > 0$ and (120). Equations (114)–(121) prove the expectation bound. Vanishing odd directional moments identify the barrier as $2b/m_0$ times $\mathbb{E}[(1-X)^{-1}]$, giving (126).

For monotonicity, differentiate (121) at the stationary node $\beta(w)$. The node-derivative term vanishes. The remaining derivative is

$$-\frac{q^2}{(1-r)(d+\beta)^2}((1-r)(1-2\alpha) + \alpha^2 + 2\beta - \frac{13}{4}q^2) \leq 0.$$

Multiplication by $2/m_0$ proves the assertion. The certificate also checks monotonicity directly after clearing the positive denominator. $\square$

Lemma 4.20 gives the following compression bound. Suppose C is admissible, $V_D \preceq \theta I$, and $(mI, 1)$ lies in the cone of H_D , with $m > 0$. For $B = bv v^*$, set

$$q_v = v^* V_D v, \quad \nu^2 = \sum_{i=1}^{N_D} (v^* D_i v)^2, \quad e = \frac{q_v - \nu^2}{m + C\sqrt{\theta}}.$$

The smaller directional eigenvalue obeys

$$\ell_B \geq \frac{b}{m - e + \sqrt{\nu^2 + e^2}}. \quad (127)$$

Actual cone membership is required, but the inequality does not require $m > C\sqrt{\theta}$. The proof in Lemma 4.20 uses the rank-two parity identity and a Schur complement, not a new root bound assumed at m .

For the complete scaled isotropic family put

$$e(C; m, s, z) = \frac{s^2(1 - z^2)}{m + Cs}, \quad \mathcal{L}(C; m, s, z) = \frac{1}{m - e(C; m, s, z) + \sqrt{s^2 z^2 + e(C; m, s, z)^2}}.$$

Theorem 4.26 (Eighth-moment schedule). *Let C, m_0, s_0 satisfy Lemma 4.25, and let k be continuous on $[s_0, 1]$. Set*

$$m(s) = m_0 + \int_{s_0}^s 2uk(u) du.$$

If, for some $\gamma > 0$, all $s \in [s_0, 1]$ and $z \in [0, 1]$ satisfy

$$k(s) \geq U_0^{(8)}(C; m_0, s_0)/2, \quad (128)$$

$$k(s) - U_z^{(8)}(C; m_0, s_0) + \mathcal{L}(C; m(s), s, z) \geq \gamma, \quad (129)$$

then $m(1)$ is admissible.

Proof. Use polynomial-level isotropic padding, so $\sum_{i=1}^n B_i^2 = I$, and omit zero matrices. Write $B_i = b_i v_i v_i^*$, and let ζ_i be its full-family directional parameter. For an integer N , set

$$\Delta = \frac{1 - s_0^2}{N}, \quad t_r = s_0^2 + r\Delta, \quad s_r = \sqrt{t_r}, \quad k_r = k(s_r), \quad m_r = m_0 + \Delta \sum_{j=1}^r k_j.$$

Start from $H_{s_0 B}$, append N copies $\sqrt{\Delta} B_i$ of each direction, and give copy (r, i) initial coordinate $-\delta_{r,i}$ with $\delta_{r,i} = k_r b_i \sqrt{\Delta}$. At spectral coordinate m_N , the initial matrix argument is exactly $m_0 I$, because $\sum_{i=1}^n b_i B_i = I$. Lemma 4.25 gives cone membership and the unused barrier bound $b_i \sqrt{\Delta} U_{\zeta_i}^{(8)}$.

Before an update in round r , let D denote the processed family and M the current matrix argument. Then

$$M \preceq m_r I, \quad (t_r - \Delta)I \preceq V_D \preceq t_r I, \quad \sum_{j=1}^{N_D} (v_j^* D_j v_j)^2 \leq t_r \zeta_i^2.$$

Cone comparison permits applying (127) at $m_r I$. Its parameter e is at least

$$e_{r,N}(z) = \frac{\max\{0, t_r(1 - z^2) - \Delta\}}{m_r + Cs_r},$$

while its ν is at most $s_r \zeta_i$. Hence the smaller eigenvalue for the copy is at least

$$\frac{b_i \sqrt{\Delta}}{m_r - e_{r,N}(\zeta_i) + \sqrt{s_r^2 \zeta_i^2 + e_{r,N}(\zeta_i)^2}}.$$

This accounts for the incomplete round. The error in e relative to a full scaled family is at most $\Delta/(m_r + Cs_r)$. Uniform Riemann-sum convergence and $m_r \geq m_0 > 0$ make the lower bound, after normalization by $b_i \sqrt{\Delta}$, converge uniformly to $\mathcal{L}(C; m(s_r), s_r, \zeta_i)$.

The strict schedule margin gives $u - \ell \leq \delta_{r,i}$ for all sufficiently large finite N . Monotonicity $U_z^{(8)} \leq U_0^{(8)}$ gives $u \leq 2\delta_{r,i}$. Lemma 2.5 therefore preserves cone membership and every unused barrier. This is an induction, not an assumption of cone membership at later steps.

Finally, in each original direction the commuting operators combine as

$$(1 - \frac{s_0^2}{2} \mathcal{D}_{B_i}^2)(1 - \frac{\Delta}{2} \mathcal{D}_{B_i}^2)^N \det(X)^2 = (1 - \frac{1}{2} \mathcal{D}_{B_i}^2) \det(X)^2,$$

because $\mathcal{D}_{B_i}^4 \det(X)^2 = 0$ and $s_0^2 + N\Delta = 1$. The identity remains true after all other commuting operators. The final polynomial is the original discrepancy polynomial. Its largest root is at most m_N for every sufficiently large N , and hence at most $m(1)$. No signing of artificial copies is selected or transferred. $\square$

We express the schedule constraints as polynomial inequalities. Use $x = (s - s_0)/(1 - s_0)$ as the schedule coordinate, and write

$$U_z^{(8)} = \frac{\mathcal{N}(z)}{\mathcal{D}(z)}, \quad \mathcal{N} = 4n_0 - n_1^2, \quad \mathcal{D} = 2m_0(1 - r)J.$$

Both are rational polynomials in z ; their degrees are eight and four. After substituting the polynomial schedule, define

$$\begin{aligned} F &= \mathcal{N} - k\mathcal{D}, \quad E = \mathcal{D} - mF, \quad h = m + Cs, \\ \mathcal{P} &= h(E^2 - s^2 z^2 F^2) + 2Es^2(1 - z^2)F. \end{aligned} \tag{130}$$

It suffices to certify $k - U_0^{(8)}/2 > 0$, $F > 0$, $E > 0$, and $\mathcal{P} > 0$. Indeed

$$\begin{aligned} h\mathcal{P} &= [hE + s^2(1 - z^2)F]^2 \\ &\quad - F^2[s^2 z^2 h^2 + s^4(1 - z^2)^2]. \end{aligned}$$

The first bracket and F are positive. Taking the positive square root gives $\mathcal{D} > F(m - e + \sqrt{s^2 z^2 + e^2})$, which is precisely the strict second schedule inequality. Compactness supplies a uniform positive margin.

We use the following six rational schedules. All terminating decimals below are exact rationals. Each schedule is

$$\begin{aligned} k(s) &= 10^{-12} \sum_{j=0}^9 a_j x^j, \quad x = \frac{s - s_0}{1 - s_0}, \\ m(s) &= m_0 + 2(1 - s_0)10^{-12} \sum_{j=0}^9 a_j \left(\frac{s_0 x^{j+1}}{j+1} + \frac{(1 - s_0)x^{j+2}}{j+2} \right). \end{aligned} \tag{131}$$

Define the successive admissible bounds by

$$(\widehat{C}_0, \dots, \widehat{C}_6) := (2.28128, 2.2563, 2.24975, 2.24790, \\ 2.24735, 2.24718, 2.24710).$$

For stages $j = 1, \dots, 6$, use $(m_{0,j}, s_{0,j})$ in the order

$$(1.7790, 0.7479), \quad (1.8065, 0.7686), \quad (1.8148, 0.7747), \\ (1.8172, 0.7765), \quad (1.8179, 0.7770), \quad (1.8182, 0.7772).$$

Stage j takes input $\widehat{C}_{j-1}$ and has endpoint below $\widehat{C}_j$.

The schedule polynomials are defined in Appendix A.1; the endpoints are given below. At each stage, rational substitution and expansion verify $m_0 > Cs_0$ and the stationary-node simplification. Exact Bernstein coefficients verify the positivity conditions in Lemma 4.25, the sign of the derivative of $U^{(8)}$, and the following inequalities on the entire square $0 \leq x, z \leq 1$:

$$k - U_0^{(8)}/2 > \frac{1}{4}, \quad F > \frac{1}{50}, \quad E > \frac{1}{100}, \quad \mathcal{P} > \frac{1}{10^7}. \quad (132)$$

The first three schedule checks use Bernstein coefficients on the whole square. The last polynomial has bidegree (51, 18) and is checked on all 256 fixed strips

$$[0, 1] \times [j/256, (j+1)/256], \quad 0 \leq j < 256.$$

For a polynomial $\sum_{i=0}^n \sum_{j=0}^m c_{ij} x^i z^j$ of bidegree (n, m) , its Bernstein coefficients on the unit square are

$$b_{ab} = \sum_{i=0}^a \sum_{j=0}^b c_{ij} \frac{\binom{a}{i}}{\binom{a}{i}} \frac{\binom{b}{j}}{\binom{b}{j}}.$$

The basis functions are nonnegative and sum to one. Exact midpoint de Casteljau subdivision gives the coefficients on the strips, so positive coefficients prove full-domain inequalities, not sampled grid tests.

Each stage checks 253255 coefficients, including the six auxiliary univariate conditions. Thus there are 1519530 new Bernstein-coefficient checks. The twenty positive Gram-matrix pivots and all directed word identities are separate checks.

The six exact endpoints are

$$E_1 = \frac{390883609484629128380501}{1732500000000000000000000} < 2.2563, \\ E_2 = \frac{194876382665291224427273}{8662500000000000000000000} < 2.24975, \\ E_3 = \frac{148355799212699961125149}{6600000000000000000000000} < 2.24790, \\ E_4 = \frac{1977610209203593669203}{8800000000000000000000000} < 2.24735, \\ E_5 = \frac{31145159951808480301841}{13860000000000000000000000} < 2.24718, \\ E_6 = \frac{64884330101504780984987}{2887500000000000000000000} < 2.24710.$$

The input 2.28128 has already been proved admissible. Six applications of Theorem 4.26 therefore give the finite chain

$$2.28128 \longrightarrow 2.2563 \longrightarrow 2.24975 \longrightarrow 2.24790 \longrightarrow 2.24735 \longrightarrow 2.24718 \longrightarrow 2.24710.$$

The last endpoint satisfies

$$\frac{22471}{10000} - \frac{64884330101504780984987}{2887500000000000000000000} = \frac{682398495219015013}{2887500000000000000000000} > 0.$$

Thus $22471/10000 = 2.24710$ is admissible; the stronger exact endpoint is admissible as well. This is the input for the tenth-moment refinement below.

4.8 The tenth-moment refinement

The tenth-moment argument follows the same sequence: a positive remainder gives a degree-five barrier, which we combine with exact rational schedules. This stage uses $0 \preceq \mathcal{E} \preceq 2I$ and the positive remainder R_3 with coefficient $2/3$ below. Lemma 4.32 later sharpens the bound to $\mathcal{E} \preceq I$ and gives R_* with coefficient $1/3$. The two remainders satisfy $R_3 = R_* + \mathcal{E}/3$. The tenth-moment identity in Eq. (138) uses R_3 , whereas the twelfth-moment Gram generators use R_* .

In this refinement normalize $\sum_{i=1}^{N_D} D_i^2 = I$, and write $D_i = w_i w_i^*$. Define

$$\begin{aligned} \Psi(X) &= \sum_{i=1}^{N_D} D_i X D_i, & W &= \sum_{i,j=1}^{N_D} \text{tr}[D_i D_j] D_i D_j, \\ X &= \text{tr}_2[(\sum_{i=1}^{N_D} D_i \otimes D_i)^3], & Y &= \sum_{i=1}^{N_D} \Psi(D_i)^2, \\ \mathcal{E} &= W + \Psi(W) - X - Y. \end{aligned}$$

Partial trace is unnormalized. Rank one gives the corresponding word expressions

$$\begin{aligned} W &= \sum_{i,j=1}^{N_D} D_i D_j D_i D_j, & X &= \sum_{i,j,k=1}^{N_D} D_i D_j D_k D_i D_j D_k, \\ Y &= \sum_{i,j,k=1}^{N_D} D_i D_j D_i D_k D_j D_k. \end{aligned}$$

All index sums are independent; there is no distinctness condition.

Lemma 4.27. *The matrices satisfy*

$$0 \preceq W \preceq I, \quad 0 \preceq \mathcal{E} \preceq 2I, \quad R_0 := Y - W^2 \succeq 0.$$

Consequently

$$R_3 := 3W^2 + \frac{2}{3}\mathcal{E} - W\mathcal{E} - \mathcal{E}W \succeq 0.$$

Proof. Put $Q_2 = \sum_{i=1}^{N_D} D_i \otimes D_i \succeq 0$. Isotropy gives

$$\sum_{i=1}^{N_D} (D_i \otimes I - I \otimes D_i)^2 = 2I - 2Q_2 \succeq 0,$$

so $Q_2 \preceq I$. Partial tracing $0 \preceq Q_2^2 \preceq Q_2$ gives

$$0 \preceq W \preceq \sum_{i=1}^{N_D} \text{tr}[D_i] D_i = \sum_{i=1}^{N_D} D_i^2 = I.$$

The map Ψ is positive and unital, hence $0 \preceq \Psi(W) \preceq I$.

For completeness recall the sixth-moment remainder identity. On $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \overline{\mathbb{C}^d}$ put

$$Lx = \sum_{j=1}^{N_D} (w_j^* x) w_j \otimes w_j \otimes \overline{w_j},$$

$$C_{1,i} = D_i \otimes I \otimes I, \quad C_{2,i} = I \otimes D_i \otimes I, \quad C_{3,i} = I \otimes I \otimes \overline{D_i}.$$

Tensor contractions give $L^*L = W$ and

$$\sum_{i=1}^{N_D} L^* C_{1,i} L D_i = Y, \quad \sum_{i=1}^{N_D} L^* C_{3,i} L D_i = X.$$

Using $\sum_{i=1}^{N_D} C_{a,i}^2 = I$ and expanding the squares yields

$$\begin{aligned} 2\mathcal{E} &= \sum_{i=1}^{N_D} (C_{1,i}L - LD_i)^* (C_{1,i}L - LD_i) \\ &\quad + \sum_{i=1}^{N_D} (C_{3,i}L - LD_i)^* (C_{3,i}L - LD_i) \succeq 0. \end{aligned}$$

Both X and Y are positive, so $\mathcal{E} \preceq W + \Psi(W) \preceq 2I$.

Define $Vx = (D_i x)_i$ and $Sx = (\Psi(D_i)x)_i$. Then $V^*V = I$, $V^*S = W$ and $S^*S = Y$. Therefore

$$R_0 = S^*(I - VV^*)S \succeq 0.$$

Finally, the identity

$$R_3 = \frac{1}{3}(3W - \mathcal{E})^2 + \frac{1}{3}\mathcal{E}(2I - \mathcal{E})$$

proves positivity. The last product is positive because both factors are functions of the same Hermitian matrix $\mathcal{E}$. $\square$

We next compute the directional moments through order ten. For a unit vector v , use the ternary determinantal representation of Section 4.1

$$H_D(xI + hvv^*, t) = \det(xI_{2d} + hP + tT), \tag{133}$$

where P is an orthogonal projection of rank two and T is real symmetric. Only this three-variable restriction is represented, and the representation may depend on v . There are Hermitian matrix expressions $\mathcal{M}_{2k}$, independent of v , such that

$$\frac{1}{2} \text{tr}[PT^{2k}] = v^* \mathcal{M}_{2k} v.$$

All odd directional moments vanish. The first expressions are

$$\mathcal{M}_2 = I, \quad \mathcal{M}_4 = 2I - W, \quad (134)$$

$$\mathcal{M}_6 = 5I - 5W - \Psi(W) + X + Y = 5I - 4W - \mathcal{E}. \quad (135)$$

The expression for $\mathcal{M}_8$ was given above. The finite construction of $\mathcal{M}_{10}$ is given in Appendix A.3.

The directional moments are computed by the following finite coefficient rule. For X near I let $G_X(i, j) = w_i^* X^{-1} w_j$. Rank one and the determinant lemma give

$$\frac{H_D(X, t)}{\det(X)^2} = \sum_{S \subseteq [N_D]} (-t^2)^{|S|} \det(G_X[S])^2. \quad (136)$$

For $|S| = k$, use ordered k -tuples of indices and divide by $k!$. Expanding the squared determinant uses two permutations $\pi, \rho \in S_k$, with sign $\text{sgn}(\pi) \text{sgn}(\rho)$. Repeated indices are allowed because their full determinant contribution vanishes. The two permutations split uniquely into their joint orbits, and their index sums factor across orbits. Thus the formal logarithm retains precisely the pairs for which the group generated by π, ρ acts transitively.

Represent $G_X(i_a, i_b)$ by a directed edge $a \rightarrow b$. Differentiating in direction vv^* removes one edge and inserts

$$-(w_{i_a}^* v)(v^* w_{i_b}),$$

which corresponds to the marked matrix $-w_{i_b} w_{i_a}^*$. At $X = I$, eliminate a self-loop using exactly

$$\sum_{i=1}^{N_D} \|w_i\|^2 w_i w_i^* = I,$$

joining adjacent edges in their original directions. The rule also applies when an adjacent edge is marked. Dummy indices are then relabeled while preserving the marked endpoints. The signed marked-edge sum is multiplied by $(-1)^{k+1}/(2k!)$.

Indeed differentiating (133) gives

$$\mathcal{D}_{vv^*} \log H_D(I, t) = \text{tr}[P(I + tT)^{-1}],$$

so this coefficient rule computes exactly the directional moments. Evenness in t proves the odd-moment assertion. For $k = 1, \dots, 5$, the transitive-pair counts are 1, 3, 26, 426, 11064. Marking an edge multiplies each count by $2k$; collecting the directed words leaves 1, 2, 5, 19, 95 classes, respectively. Evaluate this finite rule with integers and rational numbers, retaining directed complex Gram entries. An expression and its adjoint are identified only when their directed coefficients agree.

The positive tenth-moment remainder uses the following rational parameters. Write

$$g = \frac{141}{25}, \quad h = \frac{59}{5}, \quad j = \frac{59}{5}, \quad \kappa = -\frac{39}{25}.$$

The following is the main new inequality.

Theorem 4.28. *For $\sum_{i=1}^{N_D} D_i^2 = qI$, $q > 0$, the pencil (133) satisfies, with $\mu_{2k} = \frac{1}{2} \text{tr}[PT^{2k}]$,*

$$\mu_{10} - gq\mu_8 + hq^2\mu_6 - jq^3\mu_4 \leq \kappa q^5. \quad (137)$$

Equivalently,

$$25\mu_{10} - 141q\mu_8 + 295q^2\mu_6 - 295q^3\mu_4 + 39q^5 \leq 0.$$

We construct the positive expression from two Gram blocks. We first work at $q = 1$. For a triple of indices $\mathbf{i} = (i_0, i_1, i_2)$, consider every ordered quadruple $\tau = (\tau_0, \tau_1, \tau_2, \tau_3)$ of ordered lists that together contain each of 0, 1, 2 exactly once. There are $\binom{6}{3} 3! = 120$ such quadruples; denote their set by $\mathcal{T}_3$. Order them lexicographically, using the usual lexicographic order on lists. Define

$$Z_\tau(\mathbf{i}) = \left(\prod_{a \in \tau_1} D_{i_a} \right) \otimes \left(\prod_{a \in \tau_2} D_{i_a} \right) \otimes \overline{\left(\prod_{a \in \tau_3} D_{i_a} \right)} L \left(\prod_{a \in \tau_0} D_{i_a} \right).$$

Products follow the listed order, and an empty product is the identity.

For the second block, for each fixed k define five maps $S_a^{(k)}$ into the direct sum indexed by i . Their i th components are

$$\begin{aligned} S_1(i, k) &= D_k \Psi(D_i), & S_2(i, k) &= \Psi(D_i D_k), \\ S_3(i, k) &= \Psi(D_i) D_k, & S_4(i, k) &= \Psi(D_k D_i), \\ S_5(i, k) &= \Psi(D_k) D_i. \end{aligned}$$

With the isometry $Vx = (D_i x)_i$ from Lemma 4.27, put

$$\mathcal{B}_{ab} = \sum_{k=1}^{N_D} (S_a^{(k)})^* (I - VV^*) S_b^{(k)}.$$

Thus $\mathcal{B}$ is a Gram block of projected maps.

The coefficient matrices are a real symmetric 120×120 matrix Q and a real symmetric 5×5 matrix P_5 . Algorithm 1 generates their 69 rational seed coefficients. Definition A.4 constructs the matrices by polynomial factors and an exact coefficient correction. Lemma A.15 verifies their directed identity and strict positivity.

Proposition 4.29. *The matrices of Definition A.4 satisfy $Q \succ 0$, $P_5 \succ 0$, and the exact matrix identity*

$$\begin{aligned} \kappa I - \mathcal{M}_{10} + g\mathcal{M}_8 - h\mathcal{M}_6 + j\mathcal{M}_4 &= R_3 + \frac{109}{25} R_0 \\ &+ \sum_{i_0, i_1, i_2=1}^{N_D} \sum_{\tau, \sigma \in \mathcal{T}_3} Q_{\tau\sigma} Z_\tau(\mathbf{i})^* Z_\sigma(\mathbf{i}) \\ &+ \sum_{a,b=1}^5 (P_5)_{ab} \mathcal{B}_{ab}. \end{aligned} \tag{138}$$

Every exact LDL pivot of both rational matrices is greater than 10^{-6} .

Proof. Expand each tensor Gram entry using the directed contraction rule of Section 4.8. An ordinary output factor contributes an edge from the left output to the right output; the conjugate third output contributes the reversed edge. Input rows determine the marked edge. All three external indices and the two indices in the two copies of L are independent. The resulting words have at most five labels.

For the projected block expand

$$\mathcal{B}_{ab} = \sum_{i,k=1}^{N_D} S_a(i, k)^* S_b(i, k) - \sum_{k=1}^{N_D} \left(\sum_{i=1}^{N_D} D_i S_a(i, k) \right)^* \left(\sum_{i=1}^{N_D} D_i S_b(i, k) \right).$$

The five component words are, respectively,

$$(1, 2, 0, 2), (2, 0, 1, 2), (2, 0, 2, 1), (2, 1, 0, 2), (2, 1, 2, 0),$$

where 0 denotes the free index i , 1 denotes k , and 2 denotes the internal summed index in Ψ . In the first product the two internal indices are independent, while i, k are shared. In the second product k is shared, and the left and right sums over i and over the internal index are independent. This specifies the directed expansion of the projected block.

Lemma A.15 verifies coefficientwise equality in Eq. (138) for the explicit matrices of Definition A.4, together with the stated pivot bounds.

The polynomial factors before correction are bounded below by $I/500000$. Lemma A.15 bounds the correction in operator norm by 2×10^{-8} , so each corrected block has smallest eigenvalue greater than $99/50000000 > 10^{-6}$. Every successive Schur complement has the same lower bound, which proves the claimed LDL^T pivot bound. $\square$

The Gram identity now proves Theorem 4.28.

Proof. At $q = 1$, every term on the right of (138) is positive semidefinite. This follows from Lemma 4.27 and the positive Gram matrices, since $I - VV^*$ is an orthogonal projection. Taking $v^*(\cdot)v$ proves (137). For general $q > 0$, apply the normalized result to $D_i/\sqrt{q}$. Its representing matrix is $T/\sqrt{q}$; multiplying the inequality by q^5 gives the stated factors. $\square$

We use this remainder to construct a degree-five initial barrier. Normalize the full family by $\sum_{i=1}^n B_i^2 = I$. For a unit vector v put

$$z = \zeta_v = \left(\sum_{i=1}^n (v^* B_i v)^2 \right)^{1/2} \in [0, 1].$$

Let $C > 2$ be already admissible, and choose $0 < s_0 < 1$, $m_0 > Cs_0$. Set

$$q = \frac{s_0^2}{m_0^2}, \quad r = C^2 q < 1.$$

Apply the moment statements to $D_i = s_0 B_i / m_0$. The spectral measure of T^2 weighted by $P/2$ is a probability measure on $[0, r]$. If X denotes its scalar variable, then

$$\mathbb{E}[X] = q, \quad \mathbb{E}[X^2] \leq (2 - z^4)q^2, \quad (139)$$

$$\mathbb{E}[X^5] - gq\mathbb{E}[X^4] + hq^2\mathbb{E}[X^3] - jq^3\mathbb{E}[X^2] \leq \kappa q^5. \quad (140)$$

For the second inequality in (139), use (134) and

$$v^* W v \geq \left(\sum_{i=1}^{N_D} (v^* D_i v)^2 \right)^2.$$

The latter follows from $L^* L = W$ and Cauchy–Schwarz against the unit tensor $v \otimes v \otimes \bar{v}$.

Define

$$\begin{aligned} \alpha &= \frac{1 - r + gq}{2}, \\ \beta &= \frac{hq^2 - (1 - r)(1 - 2\alpha) - \alpha^2}{2}, \\ \Delta &= (1 - r)(1 - \alpha + \beta)^2, \\ H &= (1 - r)(1 - 2\alpha + \alpha^2 + 2\beta) - 2\alpha\beta + jq^3. \end{aligned}$$

The certificate will verify $\Delta > 0$ and $H > 0$. Put

$$U_0 = \frac{2}{m_0} \left(1 + q + \frac{r\beta^2 q + 2Hq^2 + \kappa q^5}{\Delta} \right), \quad (141)$$

$$U_1 = \frac{2Hq^2}{m_0\Delta}, \quad U_z^{(10)} = U_0 - U_1 z^4. \quad (142)$$

Here U_1 is the coefficient of z^4 , not the value at $z = 1$.

Lemma 4.30. *If $\Delta > 0$ and $H \geq 0$, then for every direction bvv^* ,*

$$\mathcal{D}_{bvv^*} \log H_{s_0 B}(m_0 I, 1) \leq bU_{\zeta_v}^{(10)}.$$

Initial cone membership follows from $m_0 > Cs_0$. The majorant is nonincreasing in z .

Proof. Let $p(x) = x^2 - \alpha x + \beta$ and define

$$Q_5(x) = \frac{1 + x(r - x)p(x)^2/\Delta}{1 - x}.$$

The numerator vanishes at $x = 1$, so Q_5 is a polynomial of degree five. Moreover

$$Q_5(x) - \frac{1}{1 - x} = \frac{x(r - x)p(x)^2}{\Delta(1 - x)} \geq 0 \quad (0 \leq x \leq r). \quad (143)$$

Writing $Q_5 = \sum_{i=0}^5 c_i x^i$, its coefficients satisfy

$$\begin{aligned} c_0 &= 1, & c_1 &= 1 + \frac{r\beta^2}{\Delta}, \\ c_2 &= \frac{H - jq^3}{\Delta}, & c_3 &= \frac{hq^2}{\Delta}, \\ c_4 &= -\frac{gq}{\Delta}, & c_5 &= \frac{1}{\Delta}. \end{aligned}$$

First substitute the coupled inequality (140). The coefficients of $\mathbb{E}[X^4]$ and $\mathbb{E}[X^3]$ cancel exactly. The remaining coefficient of $\mathbb{E}[X^2]$ is $H/\Delta \geq 0$, so only now substitute (139). This gives

$$\mathbb{E}[Q_5(X)] \leq 1 + q + \frac{r\beta^2 q + H(2 - z^4)q^2 + \kappa q^5}{\Delta}.$$

In particular, no independent upper bound is substituted into a negative-coefficient moment. Odd directional moments vanish, so the barrier is $2b/m_0$ times $\mathbb{E}[(1 - X)^{-1}]$. Equations (143)–(142) give the conclusion. Monotonicity follows from $U_1 \geq 0$. $\square$

Lemma 4.20 gives the following compression bound. Suppose C is admissible, $V_D \preceq \theta I$, $m > 0$, and $(mI, 1)$ lies in the cone of H_D . For bvv^* set

$$q_v = v^* V_D v, \quad \nu^2 = \sum_{i=1}^{N_D} (v^* D_i v)^2, \quad e = \frac{q_v - \nu^2}{m + C\sqrt{\theta}}.$$

The smaller directional eigenvalue obeys

$$\ell \geq \frac{b}{m - e + \sqrt{\nu^2 + e^2}}. \quad (144)$$

This uses actual cone membership, but does not require $m > C\sqrt{\theta}$. Its proof uses PTP having eigenvalues $\pm\nu$, the parity identity $\text{tr}[(PTP)(PT^2P)] = 0$, and a Schur complement. It is exactly Lemma 4.20.

For a complete scaled isotropic family define

$$e(C; m, s, z) = \frac{s^2(1 - z^2)}{m + Cs}, \quad \mathcal{L}(C; m, s, z) = \frac{1}{m - e(C; m, s, z) + \sqrt{s^2 z^2 + e(C; m, s, z)^2}}.$$

Its denominator is at least $m > 0$.

Theorem 4.31. *Suppose C, m_0, s_0 satisfy Lemma 4.30. Let k be continuous on $[s_0, 1]$ and put*

$$m(s) = m_0 + \int_{s_0}^s 2uk(u) du.$$

If, for some $\gamma > 0$, throughout $s \in [s_0, 1]$, $z \in [0, 1]$,

$$k(s) \geq U_0/2, \quad k(s) - U_z^{(10)} + \mathcal{L}(C; m(s), s, z) \geq \gamma,$$

then $m(1)$ is admissible.

Proof. Use polynomial-level padding to assume $\sum_{i=1}^n B_i^2 = I$, omit zero matrices, write $B_i = b_i v_i v_i^*$, and retain the full-family parameters $z_i = \zeta_{v_i}$. For a positive integer N , put

$$\delta_t = \frac{1 - s_0^2}{N}, \quad t_r = s_0^2 + r\delta_t, \quad s_r = \sqrt{t_r}, \quad k_r = k(s_r), \quad m_r = m_0 + \delta_t \sum_{a=1}^r k_a.$$

Start with $H_{s_0 B}$ and append N copies $\sqrt{\delta_t} B_i$ of each direction. Give copy (r, i) initial coordinate $-k_r b_i \sqrt{\delta_t}$, and set the spectral coordinate to m_N . Because $\sum_{i=1}^n b_i B_i = I$, the initial matrix argument is exactly $m_0 I$. Lemma 4.30 gives cone membership and an unused barrier at most $b_i \sqrt{\delta_t} U_{z_i}^{(10)}$.

Before an update in round r , let D be the processed family, including $s_0 B$, and let M be the current matrix argument. Then

$$M \preceq m_r I, \quad (t_r - \delta_t)I \preceq V_D \preceq t_r I, \quad \sum_{j=1}^{N_D} (v_i^* D_j v_i)^2 \leq t_r z_i^2.$$

Cone comparison permits applying (144) at $m_r I$. Its e is at least

$$e_{r,N}(z) = \frac{\max\{0, t_r(1 - z^2) - \delta_t\}}{m_r + Cs_r},$$

and its ν is at most $s_r z_i$. The function $m - e + \sqrt{\nu^2 + e^2}$ decreases in e and increases in ν , so the normalized smaller eigenvalue is at least

$$\frac{1}{m_r - e_{r,N}(z_i) + \sqrt{s_r^2 z_i^2 + e_{r,N}(z_i)^2}}.$$

This includes the incomplete-round correction. The error in e relative to a complete scaled family is at most $\delta_t/(m_r + Cs_r)$. Uniform Riemann-sum convergence and $m_r \geq m_0 > 0$ make this normalized lower bound converge uniformly to $\mathcal{L}(C; m(s_r), s_r, z_i)$.

The strictly positive schedule margin therefore gives, for sufficiently large finite N , the local-update conditions $u - \ell \leq k_r b_i \sqrt{\delta_t}$ and $u \leq 2k_r b_i \sqrt{\delta_t}$. Lemma 2.5 preserves cone membership and every unused barrier. This is an induction, not an assumed root bound at a later point.

At the end the original and copied operators combine, for each i , as

$$(1 - \frac{s_0^2}{2} \mathcal{D}_{B_i}^2)(1 - \frac{\delta_t}{2} \mathcal{D}_{B_i}^2)^N \det(X)^2 = (1 - \frac{1}{2} \mathcal{D}_{B_i}^2) \det(X)^2.$$

Indeed $\mathcal{D}_{B_i}^4 \det(X)^2 = 0$, all operators commute, and $s_0^2 + N\delta_t = 1$. Thus the final polynomial is the original one. Its largest root is at most m_N for every sufficiently large N , and therefore at most $m(1)$. No signing of the artificial copies is transferred. $\square$

We now give the rational schedules and verify their full-domain inequalities. All terminating decimals in this section are exact rational numbers. Set $x = (s - s_0)/(1 - s_0)$ and use

$$k(s) = 10^{-12} \sum_{a=0}^9 c_a x^a.$$

Writing $L = 1 - s_0$, the exact integrated schedule is

$$m(s) = m_0 + 2L10^{-12} \sum_{a=0}^9 c_a \left(\frac{s_0 x^{a+1}}{a+1} + \frac{Lx^{a+2}}{a+2} \right).$$

The exact endpoints are displayed below. Define the successive admissible bounds by

$$(\hat{C}_0, \dots, \hat{C}_7) := (2.2471, 2.23419, 2.22994, 2.22847, \\ 2.22796, 2.22778, 2.22772, 2.22767).$$

For stages $j = 1, \dots, 7$, use $(m_{0,j}, s_{0,j})$ in the order

$$(1.83625, 0.79492), \quad (1.85403, 0.80763), \quad (1.86043, 0.81213), \\ (1.86272, 0.81373), \quad (1.86353, 0.81429), \quad (1.86381, 0.81449), \\ (1.86391, 0.81455).$$

Stage j takes input $\hat{C}_{j-1}$ and has endpoint below $\hat{C}_j$.

Section 4.7 establishes admissibility of the input 2.24710. Each stage proves an endpoint below its next admissible bound. In particular no new target is assumed admissible.

We clear the square root in the compression constraint as follows. Because $U_z^{(10)} = U_0 - U_1 z^4$ is already a rational polynomial, take its denominator to be one and define

$$F = U_z^{(10)} - k, \quad E = 1 - mF, \quad h_0 = m + Cs, \\ \mathcal{P} = h_0(E^2 - s^2 z^2 F^2) + 2Es^2(1 - z^2)F. \quad (145)$$

At every stage, exact Bernstein coefficients establish on the entire square $0 \leq x, z \leq 1$ that

$$k - U_0/2 > \frac{29}{100}, \quad F > \frac{7}{25}, \quad E > \frac{4}{25}, \quad \mathcal{P} > \frac{1}{12500}. \quad (146)$$

To see the implication, put $e = s^2(1 - z^2)/h_0$. The identity

$$h_0 \mathcal{P} = [h_0 E + s^2(1 - z^2)F]^2 \\ - F^2[s^2 z^2 h_0^2 + s^4(1 - z^2)^2]$$

has a positive first bracket and $F > 0$. Taking the positive square root gives

$$E + eF > F\sqrt{s^2z^2 + e^2},$$

hence $1 > F(m - e + \sqrt{s^2z^2 + e^2})$. This is exactly $k > U_z^{(10)} - \mathcal{L}(C; m, s, z)$. Continuity and compactness supply the uniform strict margin in Theorem 4.31.

Bernstein coefficients certify the constraints throughout the domain. For a bivariate polynomial $p(x, z) = \sum_{i=0}^n \sum_{j=0}^m a_{ij} x^i z^j$ of bidegree (n, m) , its Bernstein coefficient at (u, v) is

$$\beta_{uv} = \sum_{i=0}^u \sum_{j=0}^v a_{ij} \frac{\binom{u}{i}}{\binom{n}{i}} \frac{\binom{v}{j}}{\binom{m}{j}}.$$

The Bernstein basis is nonnegative and sums to one. Thus a lower bound for every coefficient gives the same lower bound on the whole square. Exact midpoint subdivision gives the corresponding statement on subrectangles.

The first three inequalities in (146) are verified on the whole square, with bidegrees $(9, 0)$, $(9, 4)$, and $(20, 4)$. The polynomial (145) has bidegree $(51, 10)$ and is checked on all 128 fixed strips

$$[0, 1] \times [a/128, (a+1)/128], \quad 0 \leq a < 128.$$

This is complete interval coverage, not evaluation at a grid of sample points. The count per stage is

$$10 + 50 + 105 + 128 \cdot 52 \cdot 11 = 73381.$$

The seven schedules therefore use 513667 coefficient checks.

The final endpoint and strict comparison are

$$m(1) = \frac{274448663605758732532971}{123200000000000000000000},$$

$$\frac{222767}{100000} - m(1) = \frac{280394241267467029}{123200000000000000000000} > 0.$$

All seven exact endpoints are displayed below. Rational substitution and expansion also verify the scalar denominator signs and the polynomial identity in Eq. (143).

The schedule polynomials are defined in Appendix A.2. The exact endpoints are

$$\begin{aligned} E_1 &= \frac{6451139795618557044600121}{2887500000000000000000000} < 2.23419 \\ E_2 &= \frac{77266357330218247953251923}{34650000000000000000000000} < 2.22994 \\ E_3 &= \frac{28078418807645261598741229}{12600000000000000000000000} < 2.22847 \\ E_4 &= \frac{980289959892290759813699}{44000000000000000000000000} < 2.22796 \\ E_5 &= \frac{22054748034490626516094991}{99000000000000000000000000} < 2.22778 \\ E_6 &= \frac{308757679495250443303844869}{138600000000000000000000000} < 2.22772 \\ E_7 &= \frac{274448663605758732532971}{12320000000000000000000000} < 2.22767. \end{aligned}$$

Seven applications of Theorem 4.31, starting with the already proved input 2.24710, establish admissibility of

$$\frac{274448663605758732532971}{12320000000000000000000000} < \frac{222767}{100000} = 2.22767.$$

Thus 222767/100000 is admissible. This is the input for the twelfth-moment and Hessian refinements below.

The complete tenth-moment word expansion is given in Appendix A.3.

4.9 The twelfth-moment inequality

We obtain the twelfth-moment inequality by combining positive Gram expressions with an exact bound on the residual. Appendix A.4 specifies the Gram blocks.

We retain the notation of Section 4.8, writing A_4 for the matrix previously denoted X . Initially normalize $\sum_{i=1}^{N_D} D_i^2 = I$ and write $D_i = w_i w_i^*$. Define

$$\begin{aligned} \Psi(X) &= \sum_{i=1}^{N_D} D_i X D_i, & W &= \sum_{i,j=1}^{N_D} \text{tr}[D_i D_j] D_i D_j, \\ A_4 &= \text{tr}_2[(\sum_{i=1}^{N_D} D_i \otimes D_i)^3], & Y &= \sum_{i=1}^{N_D} \Psi(D_i)^2, \\ \mathcal{E} &= W + \Psi(W) - A_4 - Y. \end{aligned}$$

The partial trace is unnormalized. For a word $w = a_1 \cdots a_{2k}$ in which each of k labels appears exactly twice, set

$$T_w = \sum_{i_1, \dots, i_k=1}^{N_D} D_{i_{a_1}} \cdots D_{i_{a_{2k}}}.$$

All indices are summed independently, with no distinctness condition. In one-based notation,

$$I = T_{11}, \quad W = T_{1212}, \quad \Psi(W) = T_{123231}, \quad A_4 = T_{123123}, \quad Y = T_{121323}.$$

These identities retain the order of the complex Gram factors.

For a unit vector v , the determinantal representation theorem of Section 4.1 supplies

$$H_D(xI + uvv^*, t) = \det(xI_{2d} + u\Pi + tT), \quad (147)$$

where Π is the rank-two orthogonal projection denoted P in the earlier moment estimates, and T is real symmetric. The representation may depend on v . Define Hermitian matrix expressions $\mathcal{M}_{2k}$ by

$$\frac{1}{2} \text{tr}[\Pi T^{2k}] = v^* \mathcal{M}_{2k} v.$$

They are the coefficients of the directional logarithmic derivative, so they are independent of the choice of representation. The established identities are

$$\mathcal{M}_2 = I, \quad \mathcal{M}_4 = 2I - W, \quad \mathcal{M}_6 = 5I - 4W - \mathcal{E}. \quad (148)$$

All odd directional moments vanish, by evenness in t .

Lemma 4.32. *One has*

$$0 \preceq W \preceq I, \quad 0 \preceq \mathcal{E} \preceq I, \quad R_0 := Y - W^2 \succeq 0.$$

Also $0 \preceq A_4, Y, \Psi(W) \preceq I$, and

$$R_* := 3W^2 + \frac{1}{3}\mathcal{E} - W\mathcal{E} - \mathcal{E}W \succeq 0. \quad (149)$$

Proof. Put $Q_2 = \sum_{i=1}^{N_D} D_i \otimes D_i$. The identity

$$\sum_{i=1}^{N_D} (D_i \otimes I - I \otimes D_i)^2 = 2I - 2Q_2$$

shows $0 \preceq Q_2 \preceq I$. Partial tracing $Q_2^2 \preceq Q_2$ gives $W \preceq I$, and partial tracing $Q_2^3 \preceq Q_2$ gives $A_4 \preceq I$. Their positivity follows from these same partial-trace representations. The map Ψ is unital and completely positive. Its Schwarz inequality for Hermitian X follows directly by writing $\Psi(X) = V^*(I \otimes X)V$ with the isometry $Vx = (D_i x)_i$ and inserting the positive projection $I - VV^*$. Hence $0 \preceq \Psi(W) \preceq I$, and Schwarz gives

$$Y = \sum_{i=1}^{N_D} \Psi(D_i)^2 \preceq \sum_{i=1}^{N_D} \Psi(D_i^2) = I.$$

The positive sixth-moment remainder from Section 4.8 can also be checked directly. On $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \overline{\mathbb{C}^d}$, set

$$Lx = \sum_{i=1}^{N_D} (w_i^* x) w_i \otimes w_i \otimes \overline{w_i}, \quad C_{1,i} = D_i \otimes I \otimes I, \quad C_{3,i} = I \otimes I \otimes \overline{D_i}.$$

Tensor contractions give $L^*L = W$ and

$$2\mathcal{E} = \sum_{i=1}^{N_D} (C_{1,i}L - LD_i)^*(C_{1,i}L - LD_i) + \sum_{i=1}^{N_D} (C_{3,i}L - LD_i)^*(C_{3,i}L - LD_i).$$

Thus $\mathcal{E} \succeq 0$. For its sharper upper bound, (148) gives

$$v^*(I - \mathcal{E})v = \frac{1}{2} \text{tr}[\Pi T^2(T^2 - 2I)^2] \geq 0.$$

This is valid separately in the representation for each v , and hence proves $\mathcal{E} \preceq I$.

The maps $Vx = (D_i x)_i$ and $Sx = (\Psi(D_i)x)_i$ satisfy $V^*V = I$, $V^*S = W$, and $S^*S = Y$. Therefore $R_0 = S^*(I - VV^*)S \succeq 0$. Finally,

$$R_* = \frac{1}{3}(3W - \mathcal{E})^2 + \frac{1}{3}\mathcal{E}(I - \mathcal{E}) \succeq 0.$$

The factors in the second term commute because both are functions of $\mathcal{E}$. □

Lemma 4.33 (Pair-word contraction). *For any finite Hermitian family satisfying $\sum_{i=1}^{N_D} D_i^2 = I$, every independently summed pair word satisfies $\|T_w\| \leq 1$.*

Proof. Evaluate the word from right to left, maintaining an index register for each label encountered exactly once so far. At the first encounter of a label, create its register by the isometry $x \mapsto (D_i x)_i$. At the second encounter, contract that register by the adjoint map $(x_i)_i \mapsto \sum_{i=1}^{N_D} D_i x_i$. Permuting register positions is unitary. The resulting composition, from the original space back to itself, is exactly T_w ; each label is independently summed. Every factor has norm at most one, so the composition has norm at most one. This proof does not require rank one or positivity, only Hermitian matrices and the stated normalization. $\square$

The resulting twelfth-moment inequality is as follows.

Theorem 4.34. *Suppose $\sum_{i=1}^{N_D} D_i^2 = qI$, $q > 0$, and put $\mu_{2k} = \frac{1}{2} \text{tr}[\Pi T^{2k}]$. Then*

$$\mu_{12} - gq\mu_{10} + hq^2\mu_8 - jq^3\mu_6 + \ell q^4\mu_4 \leq \kappa q^6, \quad (150)$$

where the exact parameters are

$$\begin{aligned} g &= \frac{9471}{1250}, & h &= \frac{227047}{10000}, & j &= \frac{61293}{2000}, \\ \ell &= \frac{712934791}{100000000}, & \kappa &= -\frac{18223989}{2500000}. \end{aligned}$$

We prove the inequality by a positive Gram expression and an exact residual bound.

We compute the directed coefficients by expanding the determinant. For X near I , put $G_X(i, j) = w_i^* X^{-1} w_j$. Rank one gives

$$\frac{H_D(X, t)}{\det(X)^2} = \sum_{S \subseteq [N_D]} (-t^2)^{|S|} \det(G_X[S])^2. \quad (151)$$

For degree $2k$, use ordered k -tuples and two permutations $\pi, \rho \in S_k$. Their coefficient is

$$\frac{(-1)^k}{k!} \text{sgn}(\pi) \text{sgn}(\rho).$$

Repeated physical indices are allowed: their determinant contributions vanish after the full sum. The formal logarithm retains precisely pairs whose joint action is transitive, because index sums factor over their joint orbits.

A Gram factor is a directed edge. Differentiating an edge $a \rightarrow b$ in direction vv^* removes that edge and inserts the matrix $-w_{ib} w_{ia}^*$. A self-loop vertex is eliminated by

$$\sum_{i=1}^{N_D} \|w_i\|^2 w_i w_i^* = \sum_{i=1}^{N_D} D_i^2 = I,$$

joining its remaining adjacent edges in their original directions. The signed marked-edge sum is divided by $2k!$ and multiplied by $(-1)^{k+1}$. The resulting coefficient dictionary is exactly $\mathcal{M}_{2k}$. For $k = 1, \dots, 6$, the transitive-pair counts are 1, 3, 26, 426, 11064, 413640. Each pair contributes $2k$ marked terms, and collecting the directed words leaves 1, 2, 5, 19, 95, 588 classes, respectively. The enumeration can be reduced by taking one first permutation per conjugacy class and weighting it by the class size, since simultaneous relabeling preserves the collected marked graphs. Enumerate all second permutations for each representative.

Every residual graph is verified to have a pair-word representative. This can also be seen from an Euler trail: after removing the marked edge, start at its target and follow the directed edges to

its source. The visited vertex sequence is a pair word, with ordinary complex scalar Gram factors multiplied in any order. The marked endpoints retain the matrix factor. Compare coefficients separately for every directed graph class.

We now compare the positive Gram expression with the target moment polynomial. Let

$$\kappa_0 = -\frac{18272989}{2500000}, \quad \mathcal{T}_0 = \kappa_0 I - \mathcal{M}_{12} + g\mathcal{M}_{10} - h\mathcal{M}_8 + j\mathcal{M}_6 - \ell\mathcal{M}_4.$$

Definition A.5 gives a positive expression $\mathcal{P}_{\text{cert}}$ using the maps in Appendix A.4. Its explicit coefficients satisfy the bounds in Lemma A.15. Index the 588 possible residual classes by $w_1, \dots, w_{588}$, allowing zero coefficients, so that

$$\mathcal{T}_0 = \mathcal{P}_{\text{cert}} + \mathcal{R}, \quad \mathcal{R} = \sum_{a=1}^{588} r_{w_a} T_{w_a}. \quad (152)$$

The 31 rational Gram matrices have total coefficient dimension 810. Fifteen blocks have the polynomial form $\varepsilon I + F_b^\top F_b$ with $\varepsilon := 1/5000000000$, and the remaining sixteen equal $10^{-8}I$. All thirteen scalar weights are zero. These lower bounds imply that every $LDL^\top$ pivot exceeds 10^{-10} .

The coefficient dictionary of $\mathcal{R}$ is recomputed from (151) and the Gram definitions. It is Hermitian and satisfies the exact bound

$$\sum_{a=1}^{588} |r_{w_a}| < \frac{39}{2000} < \frac{49}{2500} =: \rho. \quad (153)$$

By Lemma 4.33,

$$\|\mathcal{R}\| \leq \sum_{a=1}^{588} |r_{w_a}| < \rho, \quad \mathcal{R} + \rho I \succeq 0.$$

Since $\mathcal{P}_{\text{cert}} \succeq 0$ and $\kappa = \kappa_0 + \rho$, equations (152)–(153) prove

$$\kappa I - \mathcal{M}_{12} + g\mathcal{M}_{10} - h\mathcal{M}_8 + j\mathcal{M}_6 - \ell\mathcal{M}_4 \succeq 0.$$

Taking the quadratic form in v proves (150) at $q = 1$. Apply this normalized result to $D_i/\sqrt{q}$ and multiply by q^6 to prove the general case. Thus neither a floating-point semidefinite residual nor a numerical eigenvalue is used as a proof inequality.

4.10 Refined compression and the initial barrier

A fourth-order Hessian identity sharpens the rank-two compression estimate. We then use the twelfth-moment inequality to construct the initial barrier.

The following identity strengthens the compression estimate without an admissible root bound on T at the current point. For an arbitrary PSD rank-one processed family, let $V = \sum_{i=1}^{N_D} D_i^2$ and use (147). On $\text{ran } \Pi$ put

$$A_0 = \Pi T \Pi, \quad M_r = \Pi T^r \Pi, \quad q_v = v^* V v, \quad \nu^2 = \sum_{i=1}^{N_D} (v^* D_i v)^2.$$

The second-order identities from Lemma 4.2 give

$$\text{tr}[A_0] = 0, \quad A_0^2 = \nu^2 I_2, \quad M_2 = q_v I_2 + R, \quad \text{tr}[R] = 0.$$

The trace and determinant of $\Pi(I + tT)^{-1}\Pi$ are both even in t : use the determinant lemma in (147). Comparing their cubic coefficients gives $\text{tr}[A_0M_2] = 0$. For traceless real symmetric 2×2 matrices this is equivalent to

$$A_0R + RA_0 = 0. \quad (154)$$

Write

$$Z = \Pi T(I - \Pi), \quad T_Q = (I - \Pi)T(I - \Pi), \quad B_0 = ZT_QZ^*.$$

Then, with $p = q_v - \nu^2$,

$$ZZ^* = pI_2 + R \succeq 0, \quad \|R\| \leq p, \quad M_3 = (2q_v - \nu^2)A_0 + B_0, \quad \text{tr}[B_0] = 0. \quad (155)$$

Lemma 4.35 (Directional Hessian identity). *Put $K = \Psi(vv^*)$, $\tau = \text{Re}(v^*VKv)$, and $b_{ij} = v^*(D_iD_j + D_jD_i)v$. Then*

$$\|R\|^2 + \text{tr}[A_0B_0] = 4(\tau - q_v\nu^2) + 2\nu^4 - \frac{1}{2} \sum_{i,j=1}^{N_D} b_{ij}^2. \quad (156)$$

In particular, if $V = q_vI$, then

$$\|R\|^2 + \text{tr}[A_0B_0] \leq 0. \quad (157)$$

If $(t - \delta)I \preceq V \preceq tI$, then the left side of (157) is at most $4t\delta$.

Proof. Set $P_v = vv^*$, $a_i = v^*D_iv$, $d_i = \text{tr}[D_i]$, and $r_{ij} = \text{tr}[D_iD_j]$. For $X = I + uP_v$ write $\lambda = u/(1 + u)$. Then

$$d_i(\lambda) = d_i - \lambda a_i, \quad r_{ij}(\lambda) = r_{ij} - \lambda b_{ij} + \lambda^2 a_i a_j.$$

The coefficient of t^4 in $\log H_D(X, t)$ is $-p_2(u)/2$, where

$$p_2 = 2 \sum_{i,j=1}^{N_D} d_i(\lambda) d_j(\lambda) r_{ij}(\lambda) - \sum_{i,j=1}^{N_D} r_{ij}(\lambda)^2.$$

Let $a = v^*V^2v$, $\phi = v^*\Psi(V)v$, and $w = v^*Wv$. Direct multiplication, using $\lambda = u - u^2 + O(u^3)$, gives

$$[u^2]p_2 = 4(a + \phi - w) + 8\tau + 2q_v^2 - \sum_{i,j=1}^{N_D} b_{ij}^2. \quad (158)$$

For detail, the coefficient of λ is $-4(a + \phi - w)$; the coefficient of λ^2 is $8\tau + 2q_v^2 - \sum_{i,j=1}^{N_D} b_{ij}^2$. The two occurrences of $\sum_{i,j=1}^{N_D} a_i a_j r_{ij}$ cancel.

On the determinantal side, $-\partial_u^2 \log H_D(I + uP_v, t)|_{u=0} = \text{tr}[(\Pi(I + tT)^{-1}\Pi)^2]$. Its coefficient of t^4 is

$$2 \text{tr}[M_4] + 2 \text{tr}[A_0M_3] + \text{tr}[M_2^2].$$

Lemma 4.7 gives $\text{tr}[M_4] = 2(a + \phi - w)$. Equations (155), $\text{tr}[A_0^2] = 2\nu^2$, and $\text{tr}[R^2] = 2\|R\|^2$ now turn equality with (158) into (156).

The real Gram matrix $G_{ij} = \text{Re}\langle D_iv, D_jv \rangle$ satisfies $G \succeq (a_i a_j)_{ij}$, by orthogonal projection onto v in the realification of the Hilbert space. Since $b_{ij} = 2G_{ij}$,

$$\sum_{i,j=1}^{N_D} b_{ij}^2 = 4 \text{tr}[G^2] \geq 4 \left(\sum_{i=1}^{N_D} a_i^2 \right)^2 = 4\nu^4.$$

Also $v^*Kv = \nu^2$. Under exact isotropy, $\tau = q_v\nu^2$, proving (157). In the approximate case, $q_v \in [t - \delta, t]$ and $\|V - q_v I\| \leq \delta$. Rank one and positivity imply $0 \preceq K \preceq V \preceq tI$, since $a_i \leq d_i$. Therefore

$$\tau - q_v\nu^2 = \operatorname{Re} v^*(V - q_v I)Kv \leq t\delta.$$

Substitution in (156) proves the final assertion. $\square$

The Hessian identity gives the following rational compression bound. All inverses of compressed matrices in the following theorem are on $\operatorname{ran} \Pi$; T_Q acts on its orthogonal complement.

Theorem 4.36. *Suppose $(mI, 1)$ is in the hyperbolicity cone of H_D , $m > \nu > 0$, and the notation above satisfies*

$$p = q_v - \nu^2 \geq 0, \quad x = \frac{p}{2m\nu} \leq 1, \quad \|R\|^2 + \operatorname{tr}[A_0 B_0] \leq \epsilon, \quad \epsilon \geq 0.$$

Define

$$\eta = \frac{p}{m} \frac{1-x}{1-x+x^2} = \frac{2\nu p(2m\nu - p)}{4m^2\nu^2 - 2m\nu p + p^2}. \quad (159)$$

For the direction bvv^* , the smaller directional eigenvalue obeys

$$\ell_{bvv^*} \geq \frac{b}{m + \nu - \eta + \epsilon/(2m^2\nu)}. \quad (160)$$

Under exact isotropy one may take $\epsilon = 0$. No inequality $m > C\sqrt{\|V\|}$ is required.

Proof. Cone membership gives $mI + T \succ 0$. Diagonalize $A_0 = \operatorname{diag}(\nu, -\nu)$. By (154), R has diagonal zero; write its off-diagonal entry as r . Since $\operatorname{tr}[B_0] = 0$, the hypothesis gives

$$(B_0)_{11} \leq \frac{\epsilon - r^2}{2\nu}.$$

The inverse compression has Schur complement

$$S = (\Pi(mI + T)^{-1}\Pi)^{-1} = mI_2 + A_0 - Z(mI + T_Q)^{-1}Z^*.$$

For any Hermitian U , completing a square gives

$$Z(mI + T_Q)^{-1}Z^* \succeq ZZ^*U + UZZ^* - UZ(mI + T_Q)Z^*U.$$

Take $U = \operatorname{diag}(1/m, 0)$ and use $Z(mI + T_Q)Z^* = mZZ^* + B_0$. This yields

$$S \preceq \begin{pmatrix} m + \nu - p/m - r^2/(2m^2\nu) & -r/m \\ -r/m & m - \nu \end{pmatrix} + \frac{\epsilon}{2m^2\nu}I_2.$$

Let $c = (1 - x)/(1 - x + x^2)$, so $\eta = 2\nu xc$. The difference between $(m + \nu - \eta)I_2$ and the first displayed matrix has lower diagonal entry $2\nu - \eta > 0$. Its determinant is at least

$$\begin{aligned} & (2\nu - \eta)(p/m - \eta) - \frac{\eta p^2}{2m^2\nu} \\ &= 4\nu^2 x((1 - c)(1 - xc) - cx^2) = \frac{4\nu^2 x^6}{(1 - x + x^2)^2} \geq 0. \end{aligned}$$

Here $r^2 \leq p^2$ was used. Thus the difference is positive semidefinite, including $p = 0$ and $x = 1$. Also $m + \nu - \eta > 0$, because $m > \nu$ and $\eta < 2\nu$. The eigenvalues in direction bvv^* are those of bS^{-1} , proving (160). $\square$

For a complete scaled isotropic family $D_i = sB_i$, put

$$\begin{aligned} \nu &= sz, \quad p = s^2(1 - z^2), \quad D = 4m^2\nu^2 - 2m\nu p + p^2, \quad N = 2\nu p(2m\nu - p), \\ J &= (m + \nu)D - N, \quad \mathcal{L}_{\text{new}}(m, s, z) = \frac{D}{J}. \end{aligned} \quad (161)$$

For $z \in [1/4, 1]$, the condition $m > 15s/8$ implies $m > \nu$ and $x < 1$. Thus $D, J > 0$ and (161) is a valid lower bound normalized by b . For $z \leq 1/4$ we instead use the simple compression from Lemma 4.2, $\ell/b \geq 1/(m + \nu) \geq 1/(m + s/4)$. This avoids a singular formula at $\nu = 0$.

We next construct the degree-six initial barrier. Normalize $\sum_{i=1}^n B_i^2 = I$, and for each unit v retain

$$z = \left(\sum_{i=1}^n (v^* B_i v)^2 \right)^{1/2} \in [0, 1].$$

Let $C > 2$ be already admissible, choose $0 < s_0 \leq 1$ and $m_0 > Cs_0$, and set

$$q = (s_0/m_0)^2, \quad r = C^2 q < 1.$$

For $D_i = s_0 B_i / m_0$, the spectral measure of T^2 weighted by $\Pi/2$ is a probability measure supported on $[0, r]$. Its scalar variable X satisfies

$$\mathbb{E}[X] = q, \quad \mathbb{E}[X^2] \leq (2 - z^4)q^2, \quad (162)$$

$$\mathbb{E}[X^6] - gq\mathbb{E}[X^5] + hq^2\mathbb{E}[X^4] - jq^3\mathbb{E}[X^3] + \ell q^4\mathbb{E}[X^2] \leq \kappa q^6. \quad (163)$$

The fourth-moment defect in (162) follows from $L^*L = W$ and Cauchy–Schwarz against $v \otimes v \otimes \bar{v}$: $v^*Wv \geq (\sum_{i=1}^{N_D} (v^* D_i v)^2)^2$.

Define

$$\begin{aligned} \alpha &= \frac{1 - r + gq}{2}, \\ \beta &= \frac{hq^2 - (1 - r)(1 - 2\alpha) - \alpha^2}{2}, \\ \gamma &= \frac{(1 - r)(1 - 2\alpha + \alpha^2 + 2\beta) - 2\alpha\beta + jq^3}{2}, \\ \Delta &= (1 - r)(1 - \alpha + \beta - \gamma)^2. \end{aligned}$$

Put $p(x) = x^3 - \alpha x^2 + \beta x - \gamma$. Define coefficients $\theta_0, \dots, \theta_6$ by

$$(1 - x) \sum_{i=0}^6 \theta_i x^i = \Delta + (r - x)p(x)^2. \quad (164)$$

The numerator vanishes at $x = 1$, so they exist. Their four highest values are

$$\theta_6 = 1, \quad \theta_5 = -gq, \quad \theta_4 = hq^2, \quad \theta_3 = -jq^3.$$

When $\Delta > 0$, the polynomial $Q_6 = \sum_{i=0}^6 \theta_i x^i / \Delta$ satisfies

$$Q_6(x) - \frac{1}{1 - x} = \frac{(r - x)p(x)^2}{\Delta(1 - x)} \geq 0 \quad (0 \leq x \leq r). \quad (165)$$

Let $H = \theta_2 - \ell q^4$. When $H \geq 0$, first applying (163) cancels the terms involving $\mathbb{E}[X^5]$, $\mathbb{E}[X^4]$, and $\mathbb{E}[X^3]$. Only then use (162). This gives

$$\mathbb{E}[Q_6(X)] \leq \frac{\theta_0 + \theta_1 q + H(2 - z^4)q^2 + \kappa q^6}{\Delta}.$$

In particular, no upper moment estimate is substituted into a coefficient of the wrong sign.

Lemma 4.37. Suppose $\Delta > 0$ and $H \geq 0$. At the initial point $(m_0 I, 1)$, every direction bvv^* has barrier at most bU_z , where

$$U_z = U_0 - U_{\text{drop}} z^4, \quad U_0 = \frac{2(\theta_0 + \theta_1 q + 2Hq^2 + \kappa q^6)}{m_0 \Delta}, \quad U_{\text{drop}} = \frac{2Hq^2}{m_0 \Delta}. \quad (166)$$

Initial cone membership follows from $m_0 > Cs_0$. The barrier majorant is nonincreasing in z .

Proof. Odd moments vanish, so the logarithmic directional derivative equals $2b/m_0$ times $\mathbb{E}[(1 - X)^{-1}]$. Apply Eq. (165) and the preceding expectation bound. Monotonicity follows from $H \geq 0$. The argument requires only $q > 0$ and $r = C^2 q < 1$, and therefore also applies at $s_0 = 1$. In that case $m_0 > C$ ensures initial cone membership by admissibility. The later finite-copy schedules still require $s_0 < 1$ to leave a positive amount of variance to process. $\square$

4.11 Final schedules and the small-direction refinement

We first combine the preceding estimates in a schedule with two directional ranges. A sharper compression for small directions then gives the final admissible bound.

Theorem 4.38. Suppose the initial parameters satisfy Lemma 4.37. Let k be continuous on $[s_0, 1]$ and put

$$m(s) = m_0 + \int_{s_0}^s 2uk(u) du.$$

Assume, uniformly over the indicated ranges, strict positive margins in

$$k(s) \geq U_0/2, \quad m(s) > 15s/8, \quad (167)$$

$$1 + (k(s) - U_0)(m(s) + s/4) > 0, \quad (168)$$

$$D + (k(s) - U_z)J > 0 \quad (1/4 \leq z \leq 1), \quad (169)$$

where D, J are from (161). Then $m(1)$ is admissible.

Proof. Polynomial-level padding reduces to $\sum_{i=1}^n B_i^2 = I$. Discard zero matrices, write $B_i = b_i v_i v_i^*$, and fix the full-family parameters $z_i = (\sum_{j=1}^n (v_i^* B_j v_i)^2)^{1/2}$. For an integer N , let

$$\delta = \frac{1 - s_0^2}{N}, \quad t_r = s_0^2 + r\delta, \quad s_r = \sqrt{t_r}, \quad k_r = k(s_r), \quad m_r = m_0 + \delta \sum_{a=1}^r k_a.$$

Start with the processed polynomial $H_{s_0 B}$ and append N copies $\sqrt{\delta} B_i$ of each direction. Set the initial coordinate of copy (r, i) to $-k_r b_i \sqrt{\delta}$ and the spectral coordinate to m_N . Since $\sum_{i=1}^n b_i B_i = I$, the initial matrix argument is exactly $m_0 I$. Lemma 4.37 gives initial cone membership and unused barrier at most $b_i \sqrt{\delta} U_{z_i}$.

Before an update in round r , let D denote all processed directions and M the current matrix argument. Completed rounds and the partial current round give

$$M \preceq m_r I, \quad (t_r - \delta)I \preceq V_D \preceq t_r I, \quad (t_r - \delta)z_i^2 \leq \nu_i^2 \leq t_r z_i^2, \quad (170)$$

where $\nu_i^2 = \sum_{j=1}^{N_D} (v_i^* D_j v_i)^2$. Cone comparison permits evaluating a lower directional eigenvalue bound at $m_r I$.

For $z_i \leq 1/4$, the simple compression gives at least $b_i \sqrt{\delta} / (m_r + s_r/4)$. For $z_i \geq 1/4$, Lemma 4.35 gives a Hessian error at most $4t_r \delta$. Also $q_i = v_i^* V_D v_i \in [t_r - \delta, t_r]$. Theorem 4.36 therefore gives the rational bound with actual $p_i = q_i - \nu_i^2$, and an additive denominator error $2t_r \delta / (m_r^2 \nu_i)$.

We spell out why this is uniform, rather than assuming exact isotropy during a partial round. Uniform Riemann-sum convergence gives $\max_r |m_r - m(s_r)| \rightarrow 0$. For all sufficiently large N , (170) and $z_i \geq 1/4$ bound ν_i away from zero uniformly. The pairs (q_i, ν_i) converge uniformly to $(t_r, s_r z_i)$. The strict margin $m(s) > 15s/8$ puts the full-family value of $p/(2m\nu)$ uniformly below one; continuity gives the same strict inequality for the actual partial-round parameters for all sufficiently large N . All rational denominators are then uniformly positive. The explicit Hessian error is $O(\delta)$ uniformly. Thus the normalized compression converges uniformly to D/J on the high-direction range.

The strict scalar margins in (168)–(169), compactness, and this convergence imply that, for sufficiently large finite N , the unused barrier u and smaller directional eigenvalue ℓ satisfy

$$u - \ell \leq k_r b_i \sqrt{\delta}, \quad u \leq 2k_r b_i \sqrt{\delta}.$$

Lemma 2.5 preserves above-root membership and all unused barrier bounds. This proves an induction; no later cone membership or new root bound is assumed.

Finally, the operators in each original direction combine because

$$(1 - \frac{s_0^2}{2} \mathcal{D}_{B_i}^2)(1 - \frac{\delta}{2} \mathcal{D}_{B_i}^2)^N \det(X)^2 = (1 - \frac{1}{2} \mathcal{D}_{B_i}^2) \det(X)^2.$$

Indeed $\mathcal{D}_{B_i}^4 \det(X)^2 = 0$, all directional derivatives commute, and $s_0^2 + N\delta = 1$. The identity remains true after the other commuting operators. The original polynomial therefore has largest root at most m_N for every sufficiently large N , hence at most $m(1)$. No signing of auxiliary copies is transferred. $\square$

The following three rational schedules make the preceding criterion explicit. Let $x = (s - s_0)/(1 - s_0)$ and $L_0 = 1 - s_0$. Each schedule is

$$k(s) = 10^{-12} \sum_{i=0}^9 a_i x^i, \tag{171}$$

$$m(s) = m_0 + 2L_0 10^{-12} \sum_{i=0}^9 a_i \left(\frac{s_0 x^{i+1}}{i+1} + \frac{L_0 x^{i+2}}{i+2} \right).$$

All terminating decimals in this construction are exact rationals. Define the successive admissible bounds by

$$(\widehat{C}_0, \dots, \widehat{C}_3) := (2.22767, 2.20750, 2.20100, 2.19868).$$

For stages $j = 1, \dots, 3$, use $(m_{0,j}, s_{0,j})$ in the order

$$(1.82709, 0.80190), \quad (1.84862, 0.81785), \quad (1.85613, 0.82333).$$

Stage j takes input $\widehat{C}_{j-1}$ and has endpoint below $\widehat{C}_j$. The schedule polynomials are defined in Appendix A.7. Compute $\Delta, H, U_0, U_{\text{drop}}$ by exact rational arithmetic. Substitution of the displayed parameters gives $\Delta, H > 0$ and $m_0 > Cs_0$ at each stage.

For the high-direction constraint, substitute $z = (1 + 3y)/4$. Exact Bernstein subdivision gives throughout $0 \leq x, y \leq 1$

$$\begin{aligned} k - U_0/2 &> 1/5, \\ m - 15s/8 &> 1/10, \\ 1 + (k - U_0)(m + s/4) &> 1/100000, \\ D + (k - U_z)J &> 1/100000. \end{aligned}$$

These are polynomial inequalities. There is no square-root squaring step in the new high-direction certificate. Their respective bidegrees are $(9, 0)$, $(11, 0)$, $(20, 0)$, and $(44, 9)$.

Convert rational power coefficients to exact Bernstein coefficients. If a cell has a nonpositive coefficient after subtracting the requested margin, replace it by both exact midpoint halves, alternating the two coordinates and starting with x at depth zero for each polynomial. For a univariate polynomial, split only the nonconstant coordinate. Use the actual degree in each coordinate and visit the lower-coordinate child first. Every accepted cell has all coefficients strictly positive. The terminal cells form an exhaustive partition of the square, not a sample grid.

The low-direction constraints use two intervals at each stage. The high-direction constraints use 19, 15, and 12 rectangles, respectively, with maximum subdivision depth six. The other two constraints are certified on the whole square. The total number of new terminal Bernstein coefficients checked is

$$20892.$$

Including the preceding stages gives 6,928,781 exact Bernstein coefficient checks. These counts exclude the moment enumeration and the polynomial Gram verification of Lemma A.15.

The three actual endpoints are

$$\begin{aligned} E_1 &= \frac{48559850328381952529141}{220000000000000000000000} < 2.20750, \\ E_2 &= \frac{52819623198521021253023}{240000000000000000000000} < 2.20100, \\ E_3 &= \frac{33859668364265216559808399}{154000000000000000000000000} < 2.19868. \end{aligned}$$

The final displayed-decimal margin is exactly

$$\frac{54967}{25000} - E_3 = \frac{3635734783440191601}{154000000000000000000000000} > 0.$$

The first input 2.22767 was established in Section 4.8. Theorem 4.38 makes E_1 admissible and hence makes 2.20750 admissible. Repeat for stages two and three. This proves admissibility of $E_3 < 2.19868$, which supplies the input for the small-direction refinement below.

A different trial matrix in the Schur complement improves the treatment of small directions. We retain the notation $A_0, R, Z, T_Q, B_0, q_v, \nu, p$ from Section 4.10, where $p = q_v - \nu^2$.

Lemma 4.39. *Suppose $(mI, 1)$ is in the cone of H_D , $m > \nu \geq 0$, $q_v > 0$, and*

$$\|R\|^2 + \text{tr}[A_0 B_0] \leq \epsilon, \quad \epsilon \geq 0.$$

Put

$$D_0 := 2m\nu + p, \quad \eta_0 := \frac{2\nu p}{D_0}.$$

Then $D_0 > 0$ and the smaller directional eigenvalue satisfies

$$\ell_{bvv^*} \geq \frac{b}{m + \nu - \eta_0 + \epsilon/(mD_0)}. \quad (172)$$

Proof. First assume $\nu > 0$ and diagonalize $A_0 = \text{diag}(\nu, -\nu)$. The anticommutation relation makes R off-diagonal; denote its real entry by r . Since $\text{tr}[B_0] = 0$, the Hessian hypothesis gives

$$(B_0)_{11} \leq \frac{\epsilon - r^2}{2\nu}.$$

The matrix $mZZ^* + B_0 = Z(mI + T_Q)Z^*$ is positive semidefinite. Its first diagonal entry therefore yields

$$r^2 \leq 2m\nu p + \epsilon. \quad (173)$$

Write $u := 2\nu/D_0$. Apply the square-completion inequality from Theorem 4.36 with $U = \text{diag}(u, 0)$. The inverse compression S satisfies

$$S \preceq M := \begin{pmatrix} m + \nu - 2pu + u^2mp + u^2(\epsilon - r^2)/(2\nu) & -ur \\ -ur & m - \nu \end{pmatrix}.$$

Let

$$K := \text{diag}(m + \nu - \eta_0 + \epsilon/(mD_0), m + \nu - \eta_0) - M.$$

Its lower diagonal entry and its Schur complement are exactly

$$K_{22} = \frac{4m\nu^2}{D_0} > 0, \\ K_{11} - \frac{K_{12}^2}{K_{22}} = \frac{p(2m\nu p - r^2 + \epsilon)}{mD_0^2} \geq 0.$$

The last inequality uses (173). Hence $K \succeq 0$, and $S \preceq (m + \nu - \eta_0 + \epsilon/(mD_0))I_2$. The denominator is positive because $\eta_0 < 2\nu$ and $m > \nu$. Taking inverses proves (172).

If $\nu = 0$, then $A_0 = 0$, $p = q_v > 0$, and the original Schur complement gives $S \preceq mI_2$. Thus $\ell_{bvv^*} \geq b/m$, which implies the claimed bound with $D_0 = p$ and $\eta_0 = 0$. This also treats the endpoint without division by ν . $\square$

For the full isotropic family scaled by s , write $\nu = sz$ and $p = s^2(1 - z^2)$. The lemma gives the rational bound

$$\mathcal{L}_0(m, s, z) := \frac{D_0}{J_0}, \quad D_0 = 2m\nu + p, \quad J_0 = (m + \nu)D_0 - 2\nu p.$$

It holds for every $z \in [0, 1]$ when $m > s > 0$. We use it on $[0, 1/2]$ and retain the stronger bound $\mathcal{L}_{\text{new}} = D/J$ from (161) on $[1/2, 1]$.

The improved compression yields six further admissibility steps. Use the same twelfth-moment initial barrier $U_z = U_0 - U_{\text{drop}}z^4$ from Lemma 4.37. Let $m(s) = m_0 + \int_{s_0}^s 2uk(u) du$. The sufficient conditions are

$$k - U_0/2 > 1/5, \quad m - 15s/8 > 1/10, \quad (174)$$

$$D_0 + (k - U_z)J_0 > 10^{-7} \quad (0 \leq z \leq 1/2), \quad (175)$$

$$D + (k - U_z)J > 10^{-7} \quad (1/2 \leq z \leq 1). \quad (176)$$

All denominators are positive by the two compression lemmas. Thus these inequalities imply the local barrier requirements with strict margins.

We justify the finite-copy passage at $z = 0$ as well. In a partial round, $(t - \delta)I \preceq V_D \preceq tI$ and the Hessian error is at most $4t\delta$. For the actual parameters, $q_v \geq t - \delta$ and $\nu^2 \leq q_v$. Since $m > \nu$,

$$D_0 = 2m\nu + q_v - \nu^2 \geq q_v \geq t - \delta.$$

Consequently $\epsilon/(mD_0) \leq 4t\delta/[m(t - \delta)]$, uniformly over the entire low-direction range. This tends to zero, and the rational formula is continuous in the actual parameters, including $\nu = 0$. On the high-direction range, ν stays uniformly positive; the preceding finite-copy proof applies unchanged.

Together with the uniform Riemann-sum estimate and the strict margins in (174)–(176), this proves the same cone induction and recombination as Theorem 4.38. Every stage therefore makes its exact endpoint $m(1)$ admissible.

The six choices below are exact terminating decimals. The first input 2.19868 was proved above; each subsequent input is the preceding output. Define the successive admissible bounds by

$$(\hat{C}_0, \dots, \hat{C}_6) := (2.19868, 2.19696, 2.19632, 2.19608, \\ 2.19599, 2.19596, 2.19595).$$

For stages $j = 1, \dots, 6$, use $(m_{0,j}, s_{0,j})$ in the order

$$(1.86463, 0.82964), \quad (1.86708, 0.83138), \quad (1.86801, 0.83204), \\ (1.86835, 0.83229), \quad (1.86851, 0.83239), \quad (1.86856, 0.83243).$$

Stage j takes input $\hat{C}_{j-1}$ and has endpoint below $\hat{C}_j$. In stage j set $x = (s - s_0)/(1 - s_0)$ and

$$k(s) = 10^{-12} \sum_{i=0}^9 a_{j,i} x^i.$$

Define $a_{j,i} := [x^i]P_j^{\text{cmp}}(x)$ using the polynomials

$$P_1^{\text{cmp}}(x) := 1030708394834 + 70877035662x - 2847694938x^2 \\ - 302459642x^3 + 52254693x^4 - 20627314x^5 \\ + 22872486x^6 - 20797207x^7 + 10884841x^8 \\ - 2409931x^9. \quad (177)$$

$$P_2^{\text{cmp}}(x) := 1031051812517 + 70142941813x - 2784934045x^2 \\ - 293753411x^3 + 44765224x^4 - 5250575x^5 \\ - 2300240x^6 + 4171852x^7 - 2635787x^8 \\ + 661352x^9. \quad (178)$$

$$P_3^{\text{cmp}}(x) := 1031195058777 + 69865776861x - 2761201333x^2 \\ - 292019033x^3 + 49175644x^4 - 19614005x^5 \\ + 22037389x^6 - 19966633x^7 + 10378012x^8 \\ - 2279810x^9. \quad (179)$$

$$P_4^{\text{cmp}}(x) := 1031290262576 + 69764980687x - 2752442247x^2 \\ - 290351674x^3 + 45659940x^4 - 10508775x^5 \\ + 6633553x^6 - 4443881x^7 + 1867130x^8 \\ - 326792x^9. \quad (180)$$

$$P_5^{\text{cmp}}(x) := 1031261083051 + 69717690089x - 2749049846x^2 \\ - 287618047x^3 + 36084043x^4 + 13537659x^5 \\ - 30515146x^6 + 29867195x^7 - 15537560x^8 \\ + 3402879x^9. \quad (181)$$

$$\begin{aligned}
P_6^{\text{cmp}}(x) := & 1031299755982 + 69703819810x - 2747255083x^2 \\
& - 290674079x^3 + 50293521x^4 - 24445111x^5 \\
& + 30815361x^6 - 29015331x^7 + 15380101x^8 \\
& - 3430336x^9.
\end{aligned} \tag{182}$$

After substituting $s = s_0 + (1 - s_0)x$, $z = y/2$ in the low range, and $z = (1 + y)/2$ in the high range, the four polynomials have bidegrees $(9, 0)$, $(11, 0)$, $(32, 7)$, and $(44, 9)$. Exact Bernstein subdivision certifies Eqs. (174)–(176) with 271266 positive terminal coefficients. Each rejected rectangle is replaced by both midpoint halves, so the terminal rectangles have disjoint interiors and cover the unit square.

The final exact endpoint is

$$E_6 = \frac{38044731798005302244190817}{17325000000000000000000000} < \frac{43919}{20000} = 2.19595.$$

The exact difference is

$$\frac{43919}{20000} - E_6 = \frac{101951994697755809183}{17325000000000000000000000} > 0.$$

Thus $43919/20000$ is admissible. The full chain through this stage uses 7200047 exact Bernstein coefficient checks. The new checks use the previously proved twelfth-moment inequality and do not assume a new moment inequality or a smaller spectral support bound.

4.12 A resolvent refinement

4.12.1 Directional spectral measures

Normalize a finite family of positive semidefinite rank-one matrices B_i by

$$\sum_{i=1}^n B_i^2 = I.$$

Define the homogeneous polynomial

$$H_B(X, t) = \prod_{i=1}^n \left(1 - \frac{t^2}{2} \mathcal{D}_{B_i}^2\right) \det(X)^2.$$

A number C is admissible if the largest zero of $H_B(xI, 1)$ is at most C for every such normalized family. By the padding argument in Section 4.3, it suffices to work with this equality normalization.

Fix one unit vector v . The ternary representation of Lemma 4.2 supplies a rank-two orthogonal projection P and a real symmetric matrix T such that

$$H_B(xI + uvv^*, t) = \det(xI_{2d} + uP + tT). \tag{183}$$

The representation may depend on B and v . For this fixed pair, it is kept fixed throughout all values of x , u , and t below. In particular, the spectral measure used at different scalar arguments is the same measure.

If C is admissible, $\|T\| \leq C$. The measure of $Y = T^2$ weighted by $P/2$ is a probability measure supported on $[0, C^2]$. All odd P -weighted moments of T vanish. Define

$$g_v(a) = \mathcal{D}_{vv^*} \log H_B(aI, 1) = \text{tr}[P(aI + T)^{-1}] = \mathbb{E}\left[\frac{2a}{a^2 - Y}\right], \quad a > C. \tag{184}$$

Differentiation under this finite spectral sum gives

$$-g'_v(a) = \mathbb{E}\left[\frac{2(a^2 + Y)}{(a^2 - Y)^2}\right]. \quad (185)$$

The normalized direction parameter is

$$z_v = \left(\sum_{i=1}^n (v^* B_i v)^2\right)^{1/2} \in [0, 1].$$

Coupled moment inequalities Set

$$\begin{aligned} p_2(y) &= y^2, \\ p_3(y) &= y^3 - 4y^2, \\ p_4(y) &= y^4 - 4y^3 + \frac{13}{4}y^2, \\ p_5(y) &= y^5 - \frac{141}{25}y^4 + \frac{59}{5}y^3 - \frac{59}{5}y^2, \\ p_6(y) &= y^6 - gy^5 + hy^4 - jy^3 + \ell y^2, \end{aligned}$$

where

$$g = \frac{9471}{1250}, \quad h = \frac{227047}{10000}, \quad j = \frac{61293}{2000}, \quad \ell = \frac{712934791}{100000000}, \quad \kappa = -\frac{18223989}{2500000}.$$

Lemma 4.12, Lemma 4.18, and the eighth-, tenth-, and twelfth-moment inequalities above imply

$$\begin{aligned} \mathbb{E}[Y] &= 1, & \mathbb{E}[p_2(Y)] &\leq 2 - z_v^4, \\ \mathbb{E}[p_3(Y)] &\leq -3, & \mathbb{E}[p_4(Y)] &\leq \frac{1}{2}, & \mathbb{E}[p_5(Y)] &\leq -\frac{39}{25}, & \mathbb{E}[p_6(Y)] &\leq \kappa. \end{aligned} \quad (186)$$

These are coupled inequalities: individual moments multiplying negative coefficients are not independently replaced by upper bounds.

4.12.2 Compression functions

For $m > 15s/8$ and $s > 0$, put

$$\nu = sz, \quad p = s^2(1 - z^2).$$

The rational compression functions from Lemma 4.39 and Theorem 4.36 are

$$D_0 = 2m\nu + p, \quad J_0 = (m + \nu)D_0 - 2\nu p, \quad L_0(m, s, z) = \frac{D_0}{J_0}, \quad (187)$$

$$D_1 = 4m^2\nu^2 - 2m\nu p + p^2, \quad J_1 = (m + \nu)D_1 - 2\nu p(2m\nu - p), \quad L_1(m, s, z) = \frac{D_1}{J_1}. \quad (188)$$

All displayed denominators are positive on the ranges where we use them. For a full isotropic family scaled by s , at a point $(mI, 1)$ in the actual cone, the smaller directional eigenvalue in direction bvv^* is at least

$$bL_0(m, s, z_v) \quad (0 \leq z_v \leq 1/2), \quad bL_1(m, s, z_v) \quad (1/2 \leq z_v \leq 1).$$

For L_1 , the compression condition $p/(2m\nu) < 1$ follows on this high-direction range from $m > 15s/8$. At $z = 0$, $L_0(m, s, 0) = 1/m$, so the low-direction formula has no singularity. These estimates require actual cone membership, but do not require $m > Cs$.

4.12.3 The infinitesimal inequality

Lemma 4.40 (One-sided complete-round resolvent inequality). *Let C be admissible and $a > C$, $a > 15/8$. Suppose A, B with $B \geq 0$ satisfy, for every normalized family and every unit vector v ,*

$$g_v(a) \leq A - Bz_v^4. \quad (189)$$

Suppose also that a scalar K satisfies

$$\begin{aligned} K &\geq A/2, \\ K &\geq A - Bz^4 - L_0(a, 1, z) \quad (0 \leq z \leq 1/2), \\ K &\geq A - Bz^4 - L_1(a, 1, z) \quad (1/2 \leq z \leq 1). \end{aligned} \quad (190)$$

Then every directional measure just defined satisfies

$$(a - 2K)(-g'_v(a)) \leq g_v(a), \quad (191)$$

or, equivalently,

$$\mathbb{E}[R_{a,K}(Y)] \leq 0, \quad R_{a,K}(y) = \frac{(a - K)y - Ka^2}{(a^2 - y)^2}. \quad (192)$$

Proof. Fix one normalized finite family and one arbitrary test vector v . Write $B_i = b_i v_i v_i^*$, with $b_i > 0$ and unit vectors v_i , after discarding zero matrices. Rank one gives

$$\sum_{i=1}^n b_i B_i = \sum_{i=1}^n B_i^2 = I.$$

Fix $\eta > 0$. Starting from the already processed polynomial H_B , append one new copy $\sqrt{\delta}B_i$ for every original direction, where $\delta > 0$ will tend to zero. Give this copy the shift

$$\Delta_i = (K + \eta)b_i\sqrt{\delta}.$$

Choose initial coordinates so the initial matrix argument is aI . After all copies have been shifted, the matrix argument is exactly

$$aI + \sum_{i=1}^n \Delta_i \sqrt{\delta} B_i = (a + (K + \eta)\delta)I.$$

Add an auxiliary coordinate in direction vv^* and leave that coordinate unprocessed and fixed at zero throughout. Its barrier is the quantity we compare.

At the initial point, the barrier for the copy of B_i is at most

$$b_i\sqrt{\delta}(A - Bz_{v_i}^4).$$

The smaller directional eigenvalue for that copy, divided by $b_i\sqrt{\delta}$, is bounded below by the appropriate L_0 or L_1 . Thus (190), with $K + \eta$ in place of K , gives strict slack in both local-update conditions from Lemma 2.5: $u \leq 2\Delta_i$ and $u - \lambda_{\min} \leq \Delta_i$.

For sufficiently small δ , these same strict conditions hold at every intermediate step. Here the family is fixed and has finitely many directions. The current polynomial coefficients and matrix argument converge to those of H_B at aI as $\delta \downarrow 0$. The relevant one-variable directional polynomials, their logarithmic derivatives, and their two directional eigenvalues therefore converge as well.

The initial point has a positive root gap since $a > C$. No continuous choice of a determinantal representation is needed: the directional eigenvalues are determined by the corresponding scalar directional polynomial. Applying the local update inductively proves cone membership and maintains all unused barriers, including the auxiliary test barrier. Uniformity over dimensions or family sizes is not required for this pointwise limiting argument.

The new and old operators recombine exactly. Indeed, $\mathcal{D}_{B_i}^4 \det(X)^2 = 0$, all directional derivatives commute, and therefore

$$(1 - \frac{1}{2}\mathcal{D}_{B_i}^2)(1 - \frac{\delta}{2}\mathcal{D}_{B_i}^2) \det(X)^2 = (1 - \frac{1+\delta}{2}\mathcal{D}_{B_i}^2) \det(X)^2.$$

The identity remains true after the operators in the other directions have been applied. Consequently the final processed family is exactly $\sqrt{1+\delta} B$.

Define

$$F_v(m, t) = \mathcal{D}_{vv^*} \log H_{\sqrt{t}B}(mI, 1).$$

Homogeneity gives

$$F_v(m, t) = t^{-1/2} g_v(m/\sqrt{t}).$$

The monotonicity of the auxiliary barrier in the finite construction proves the genuine one-sided comparison

$$F_v(a + (K + \eta)\delta, 1 + \delta) \leq F_v(a, 1) \quad (193)$$

for all sufficiently small positive δ . Both sides agree at $\delta = 0$. Differentiating this one-sided comparison at zero yields

$$(K + \eta - \frac{a}{2})g'_v(a) - \frac{1}{2}g_v(a) \leq 0.$$

Letting $\eta \downarrow 0$ proves (191). This differentiates an actual finite-round comparison, not the upper bound (189).

Finally, (184) and (185) give the exact identity

$$(a - 2K)(-g'_v(a)) - g_v(a) = 4\mathbb{E}\left[\frac{(a - K)Y - Ka^2}{(a^2 - Y)^2}\right],$$

which proves (192). □

Remark 4.41. The conclusion is a new constraint on the same directional spectral measure that already satisfies the polynomial moment inequalities. It remains valid when a later step has proved a smaller admissible constant. All 20 chosen pairs in the certificate also satisfy $a - 2K > 0$; their minimum is 45639/500000. Positivity of this quantity is useful but is not an extra hypothesis of the lemma.

4.12.4 Rational initial barriers

Proposition 4.42 (Rational majorant). *Suppose the previously proved constraints are*

$$\mathbb{E}[R_{a_r, K_r}(Y)] \leq 0 \quad (r \in \mathcal{J}).$$

Let $a > C$ and $a_r > C$ for all used references. Suppose $c_2, \dots, c_6$ and d_r are nonnegative rationals, while λ_0, λ_1 are arbitrary rationals, and the following pointwise inequality holds on the entire interval $0 \leq y \leq C^2$:

$$\frac{2a}{a^2 - y} \leq \lambda_0 + \lambda_1 y + \sum_{k=2}^6 c_k p_k(y) + \sum_{r \in \mathcal{J}} d_r R_{a_r, K_r}(y). \quad (194)$$

Then (189) holds with

$$\begin{aligned} A &= \lambda_0 + \lambda_1 + 2c_2 - 3c_3 + \frac{1}{2}c_4 - \frac{39}{25}c_5 + \kappa c_6, \\ B &= c_2. \end{aligned} \tag{195}$$

Proof. Integrate (194) against the spectral measure. Use $\mathbb{E}[Y] = 1$, the coupled inequalities (186), and the previously proved rational constraints. The multiplier signs justify every inequality. The only direction-dependent bound is $\mathbb{E}[p_2(Y)] \leq 2 - z_v^4$, giving exactly (195). $\square$

To check (194) exactly, multiply its right side minus its left side by

$$(a^2 - y) \prod_{r \in \mathcal{J}} (a_r^2 - y)^2. \tag{196}$$

This denominator is strictly positive throughout the full interval. The resulting polynomial has rational coefficients. Proving that polynomial strictly positive suffices. The exact Bernstein procedure described below certifies this positivity from the rational majorants in Appendix A.8. A rational constraint is inserted only after its own initial-barrier and K conditions have been proved. All references point to earlier entries, so the process is not circular.

The initial barrier The first pair is initialized from the $s_0 = 1, m_0 = a > C$ case of Lemma 4.37, whose proof includes this endpoint. Equivalently, for any $0 < s < 1$, homogeneity gives

$$\mathcal{D}_{bvv^*} \log H_B(aI, 1) = s \mathcal{D}_{bvv^*} \log H_{sB}(saI, 1),$$

so the same unit-variance estimate follows by applying that lemma at $(m_0, s_0) = (sa, s)$. For completeness its scalar formula is recorded here. At unit variance put

$$\begin{aligned} q &= a^{-2}, & r &= C^2 q, \\ \alpha &= \frac{1 - r + gq}{2}, & \beta &= \frac{hq^2 - (1 - r)(1 - 2\alpha) - \alpha^2}{2}, \\ \gamma &= \frac{(1 - r)(1 - 2\alpha + \alpha^2 + 2\beta) - 2\alpha\beta + jq^3}{2}, & \Delta &= (1 - r)(1 - \alpha + \beta - \gamma)^2. \end{aligned}$$

Let $p(x) = x^3 - \alpha x^2 + \beta x - \gamma$, and define θ_i by polynomial division:

$$(1 - x) \sum_{i=0}^6 \theta_i x^i = \Delta + (r - x)p(x)^2.$$

The numerator vanishes at $x = 1$, and

$$\theta_6 = 1, \quad \theta_5 = -gq, \quad \theta_4 = hq^2, \quad \theta_3 = -jq^3.$$

With $H = \theta_2 - \ell q^4$, provided $\Delta > 0$ and $H \geq 0$, the initial bound is

$$A = \frac{2(\theta_0 + \theta_1 q + 2Hq^2 + \kappa q^6)}{a\Delta}, \quad B = \frac{2Hq^2}{a\Delta}. \tag{197}$$

The certificate uses the input $C = 43919/20000$ and the initial pair

$$(a, K) = \left(\frac{56}{25}, \frac{217389}{250000}\right).$$

Exact rational evaluation of Eq. (197) gives the required signs. Bernstein subdivision certifies the two K inequalities on their full intervals, so Lemma 4.40 applies.

4.12.5 Finite schedules and transfer of the new barrier

Lemma 4.43 (Schedule for a rationally improved initial barrier). *Suppose C is admissible, $0 < s_0 < 1$, $m_0 > Cs_0$, and*

$$g_v(m_0/s_0) \leq A - Bz_v^4 \quad \text{for every normalized family and unit } v, \quad B \geq 0.$$

Let k be continuous and set

$$m(s) = m_0 + \int_{s_0}^s 2u k(u) du, \quad U_z = \frac{A - Bz^4}{s_0}.$$

Suppose the following inequalities have uniform strict positive margins:

$$k(s) > \frac{A}{2s_0}, \quad m(s) > \frac{15}{8}s, \quad s_0 \leq s \leq 1, \quad (198)$$

$$D_0 + (k(s) - U_z)J_0 > 0, \quad s_0 \leq s \leq 1, \quad 0 \leq z \leq 1/2, \quad (199)$$

$$D_1 + (k(s) - U_z)J_1 > 0, \quad s_0 \leq s \leq 1, \quad 1/2 \leq z \leq 1. \quad (200)$$

Here the compression polynomials are evaluated at $(m(s), s, z)$. Then $m(1)$ is admissible.

Proof. The scaling identity used in Lemma 4.40 gives the initial unused directional barrier for bvv^* in $H_{s_0B}(m_0I, 1)$ as at most bU_{z_v} . This replaces the original degree-six initial barrier in Section 4.11; no other part of its finite-copy argument changes.

For clarity, choose N rounds with $\delta = (1 - s_0^2)/N$, $t_r = s_0^2 + r\delta$, $s_r = \sqrt{t_r}$, and $k_r = k(s_r)$. Give each remaining copy $\sqrt{\delta}B_i$ the shift $k_rb_i\sqrt{\delta}$ and use

$$m_r = m_0 + \delta \sum_{h=1}^r k_h.$$

Before an update in round r , the current matrix is at most m_rI , and the processed variance is between $(t_r - \delta)I$ and t_rI . The current squared direction parameter is between $(t_r - \delta)z_i^2$ and $t_rz_i^2$. Cone comparison permits estimating the smaller directional eigenvalue at m_rI .

The Hessian error in Lemma 4.35 is at most $4t_r\delta$. On the low-direction range, its additive denominator error is bounded by

$$\frac{4t_r\delta}{m_r(t_r - \delta)},$$

because $2m_r\nu + q_v - \nu^2 \geq q_v \geq t_r - \delta$. This includes $z_i = 0$. On the high-direction range ν stays uniformly bounded away from zero, and Theorem 4.36 has a uniformly vanishing error as well. Uniform Riemann-sum convergence and $s_0 > 0$ therefore show that the normalized smaller eigenvalue bounds converge uniformly to L_0 and L_1 .

The positive denominators and strict margins in (199)–(200) give $k_r > U_{z_i} - L_i$ for sufficiently large N , with the small incomplete-round errors included. Since $U_{z_i} \leq A/s_0$, (198) also gives the half-barrier condition. Lemma 2.5 now preserves actual cone membership and every remaining barrier at every step. This is an induction, not an assumption of a root bound at the later scalar argument.

For each original direction the operators recombine exactly:

$$(1 - \frac{s_0^2}{2}\mathcal{D}_{B_i}^2)(1 - \frac{\delta}{2}\mathcal{D}_{B_i}^2)^N \det(X)^2 = (1 - \frac{1}{2}\mathcal{D}_{B_i}^2) \det(X)^2.$$

Thus the original polynomial has largest root at most m_N for every sufficiently large N . Passing to the limit yields $m(1)$. No signing of the artificial copies is chosen or transferred. $\square$

4.12.6 Exact data and verification results

Two admissibility stages All terminating decimals in this section denote exact rational numbers. Define the successive admissible bounds for the two stages by

$$(\widehat{C}_0, \dots, \widehat{C}_2) := (2.19595, 2.1812, 2.17969).$$

For stages $j = 1, \dots, 2$, use $(m_{0,j}, s_{0,j})$ in the order

$$(1.97233, 0.89727), \quad (1.98644, 0.90542).$$

Stage j takes input $\widehat{C}_{j-1}$ and has endpoint below $\widehat{C}_j$. Stage 1 uses the first 12 proved rational constraints. After this stage lowers the admissible constant, eight further rational constraints are certified, bringing the total to 20. Stage 2 uses only constraints already proved at that point.

The initial normalized barriers $A - Bz^4$ for these stages are exactly

$$\begin{aligned} A_1 &= \frac{3389320559923576023}{25000000000000000000}, & B_1 &= \frac{20273029467}{125000000000}, \\ A_2 &= \frac{3422060876588150943}{25000000000000000000}, & B_2 &= \frac{84893907553}{500000000000}. \end{aligned}$$

For each stage put $x = (s - s_0)/(1 - s_0)$ and

$$\begin{aligned} k(s) &= 10^{-12} \sum_{i=0}^9 c_i x^i, \\ m(s) &= m_0 + 2(1 - s_0) 10^{-12} \sum_{i=0}^9 c_i \left(\frac{s_0 x^{i+1}}{i+1} + \frac{(1 - s_0) x^{i+2}}{i+2} \right). \end{aligned}$$

In stage j , set $c_i := [x^i] P_j^{\text{res}}(x)$, where

$$\begin{aligned} P_1^{\text{res}}(x) &:= 1050714784359 + 41729993612x - 1123868400x^2 \\ &\quad - 57245704x^3 + 4814322x^4 - 324760x^5 \\ &\quad + 45600x^6 - 743x^7 - 396x^8 \\ &\quad - 104x^9. \end{aligned} \tag{201}$$

$$\begin{aligned} P_2^{\text{res}}(x) &:= 1053102032341 + 38420945216x - 945995436x^2 \\ &\quad - 46879946x^3 + 3472370x^4 - 212914x^5 \\ &\quad + 73346x^6 - 48246x^7 + 25592x^8 \\ &\quad - 5853x^9. \end{aligned} \tag{202}$$

The actual endpoints, stronger than the advertised outputs, are

$$\begin{aligned} E_1 &= \frac{302312050548252071798800643}{1386000000000000000000000000} < \frac{5453}{2500}, \\ E_2 &= \frac{18881507341815676610321299}{866250000000000000000000000} < \frac{217969}{100000}. \end{aligned}$$

The exact strict margins are

$$\begin{aligned} \frac{5453}{2500} - E_1 &= \frac{2269451747928201199357}{1386000000000000000000000000}, \\ \frac{217969}{100000} - E_2 &= \frac{57283184323389678701}{866250000000000000000000000}. \end{aligned}$$

Both numerators are positive integers.

What the finite checks establish Appendix A.8 prints the seed parameters, all subsequent rational majorants, their nonnegative multipliers, and the two initial barriers. Indices are zero-based. Entry 0 uses Eq. (197); entries 1, ..., 19 use Eq. (194). The schedule coefficients and endpoints are displayed above.

For a polynomial $p(x) = \sum_{i=0}^n a_i x^i$, its Bernstein coefficients on $[0, 1]$ are

$$b_k = \sum_{i=0}^k a_i \frac{\binom{k}{i}}{\binom{n}{i}}.$$

The Bernstein basis is nonnegative and sums to one. Strict positivity of every coefficient proves strict positivity on the full interval. Affine substitution handles other intervals. The two-variable check applies the same transformation in both variables. If some coefficients are nonpositive, bisect the interval or rectangle at its midpoint and apply the same test to *both* halves. Only strictly positive terminal coefficient arrays are accepted. This produces full-domain certificates, not sampled-grid evidence.

For the schedules, use $s = s_0 + (1 - s_0)x$, and $z = y/2$ or $z = (1 + y)/2$ on the two direction ranges. The four polynomials in (198)–(200) have respective bidegrees

$$(9, 0), \quad (11, 0), \quad (32, 7), \quad (44, 9).$$

Exact rational evaluation checks the parameter signs and backward references. The Bernstein procedure establishes the cleared polynomial inequalities, and exact differentiation and substitution verify the antiderivatives and endpoints. The twenty rational constraints and two schedules pass exact verification. The calculation accepts 680 cells with 76176 positive terminal Bernstein coefficients and maximum subdivision depth fourteen. The resulting admissible bound is 217969/100000.

Starting from the admissible value 43919/20000 established in Section 4.11, the seed barrier initializes Lemma 4.40. Proposition 4.42 and Lemma 4.40 then establish the first 12 constraints in order. Lemma 4.43 makes $E_1 < 2.1812$ admissible. The next eight constraints use this smaller admissible constant, and the second schedule makes

$$E_2 = \frac{18881507341815676610321299}{86625000000000000000000000} < \frac{217969}{100000} = 2.17969$$

admissible. This supplies the input for the further refinement below.

4.13 A further moment and resolvent refinement

We combine the resolvent inequalities of Section 4.12 with another twelfth-moment inequality. The input is the already admissible constant $C := 217969/100000$. The new moment inequality is used together with the earlier inequalities; it need not dominate them coefficientwise.

Lemma 4.44. *Under the hypotheses of Theorem 4.34, Eq. (150) holds with*

$$\begin{aligned} g_n &:= \frac{9763988689}{12500000000}, & h_n &:= \frac{230610249759}{10000000000}, \\ j_n &:= \frac{150180210121}{5000000000}, & \ell_n &:= \frac{19876891039}{2000000000}, & \kappa_n &:= -\frac{16508788487}{5000000000}. \end{aligned}$$

In particular, the normalized spectral measure in Eq. (184) satisfies

$$\mathbb{E}[p_6^n(Y)] \leq \kappa_n, \quad p_6^n(y) := y^6 - g_n y^5 + h_n y^4 - j_n y^3 + \ell_n y^2.$$

Proof. Normalize $q = 1$. Use the directed moment expansion and the positive Gram maps of Appendix A.4. The thirteen blocks are ordered by the construction below. All other block coefficients are zero. First take the `tensor4` blocks in the order `sign`, `standard`, `sign-twisted standard`, and `pair`, with dimensions 35, 105, 105, 70. Next take `project_i_k.1` and `project_i`, with dimensions 36, 15. Then take the `norm4` blocks in the order `trivial`, `sign`, `standard`, `sign-twisted standard`, and `pair`, with dimensions 15, 15, 45, 45, 30. Finally take the `trivial` and `sign norm2` blocks, both of dimension 45. The total coefficient dimension is 606. For each block $b \in [13]$, Definition A.6 constructs a square rational polynomial factor F_b of order n_b and sets $Q_b := F_b^\top F_b \succeq 0$. The state ordering, tensor conjugation, group averaging, and projection conventions are those of Appendix A.4. Thus their sum $\mathcal{P} := \mathcal{P}_F$ is positive semidefinite.

Put $\kappa_0 := -16510788487/5000000000$ and $\rho := 1/2500$. The exact directed expansion gives

$$\kappa_0 I - \mathcal{M}_{12} + g_n \mathcal{M}_{10} - h_n \mathcal{M}_8 + j_n \mathcal{M}_6 - \ell_n \mathcal{M}_4 = \mathcal{P} + \mathcal{R}.$$

The Hermitian residual is supported on 588 directed pair-word classes, and its coefficients satisfy

$$\sum_{a=1}^{588} |r_a| < \frac{3}{8000} < \rho.$$

Lemma A.15 verifies this residual bound for the explicit factors of Definition A.6. Its finite check reconstructs the moments from the transitive permutation expansion and encloses the polynomial Gram factors by exact rational arithmetic. At order twelve the enumeration contains 413640 transitive permutation pairs and 4963680 marked terms. Lemma 4.33 gives $\|\mathcal{R}\| < \rho$. Since $\kappa_n = \kappa_0 + \rho$, adding ρI to the identity proves the required positive semidefinite inequality. Taking the quadratic form in v and rescaling by q^6 proves the claim. $\square$

Six additional resolvent constraints Keep the zero-based indices $0, \dots, 19$ of the preceding certificate. For a previously proved pair (a_r, K_r) , write

$$R_{a_r, K_r}(y) := \frac{(a_r - K_r)y - K_r a_r^2}{(a_r^2 - y)^2}.$$

We use majorants of the form

$$\frac{2a}{a^2 - y} \leq \lambda_0 + \lambda_1 y + c_2 y^2 + c_n p_6^n(y) + d R_{a_r, K_r}(y), \quad 0 \leq y \leq C^2. \quad (203)$$

Here $c_2, c_n, d \geq 0$. Integrating against the same spectral measure, Lemma 4.44 and the previously established constraint give

$$g_v(a) \leq A - B z_v^4, \quad A := \lambda_0 + \lambda_1 + 2c_2 + \kappa_n c_n, \quad B := c_2. \quad (204)$$

This is the argument of Proposition 4.42 with one additional valid moment inequality.

For $j = 20, \dots, 25$, define the arguments and bounds by

$$(a_{20}, \dots, a_{25}) := 10^{-3}(2183, 2190, 2186, 2182, 2185, 2182).$$

$$(K_{20}, \dots, K_{25}) := 10^{-6}(1011899, 960109, 978910, 1000527, 977054, 993245).$$

At step j , use the previously proved constraint with index $r = j - 1$. All terminating decimals in this subsection denote exact rationals. Use the rational majorant $10^{-12}Q_j(y)$ in Eq. (203), where

$$Q_{20}(y) := 982818910112 + 138262053875y + 185447098131y^2 + 18095072787p_6^n(y) + 297313668043R_{a_{19},K_{19}}(y). \quad (205)$$

$$Q_{21}(y) := 944733207826 + 133093513316y + 179131938082y^2 + 17581385430p_6^n(y) + 129217239184R_{a_{20},K_{20}}(y). \quad (206)$$

$$Q_{22}(y) := 973587890419 + 131317412162y + 168495127560y^2 + 14144401909p_6^n(y) + 263382219679R_{a_{21},K_{21}}(y). \quad (207)$$

$$Q_{23}(y) := 983731634193 + 134254632051y + 176398338736y^2 + 15632679623p_6^n(y) + 300962458769R_{a_{22},K_{22}}(y). \quad (208)$$

$$Q_{24}(y) := 956898932018 + 134071408365y + 181865873318y^2 + 17674448129p_6^n(y) + 177440046080R_{a_{23},K_{23}}(y). \quad (209)$$

$$Q_{25}(y) := 977280304427 + 133521491030y + 176587771965y^2 + 15765039041p_6^n(y) + 270783366107R_{a_{24},K_{24}}(y). \quad (210)$$

At every step, $a > C$ and $a_r > C$. Multiplying the difference between the right and left sides of Eq. (203) by $(a^2 - y)(a_r^2 - y)^2$ gives a degree-nine polynomial. Exact Bernstein subdivision proves it strictly positive on $[0, C^2]$. The same certificate proves

$$K > A/2, \quad D_0 + (K - A + Bz^4)J_0 > 0 \quad (0 \leq z \leq 1/2),$$

and

$$D_1 + (K - A + Bz^4)J_1 > 0 \quad (1/2 \leq z \leq 1),$$

where D_i, J_i are the compression polynomials of Section 4.12, evaluated at $(m, s, z) = (a, 1, z)$. Their denominators are positive. Lemma 4.40 therefore gives $\mathbb{E}[R_{a,K}(Y)] \leq 0$ at each step in order. Every reference points backward, so the repeated value $a = 2.182$ introduces no circular dependence.

The final initial barrier In Eq. (203), take

$$a := \frac{4373}{2000}, \quad a_r := \frac{1091}{500}, \quad K_r := \frac{198649}{200000},$$

using constraint 25, and set

$$\begin{aligned} \lambda_0 &:= \frac{95090672613}{100000000000}, & \lambda_1 &:= \frac{26350501399}{200000000000}, \\ c_2 &:= \frac{5546772823}{31250000000}, & c_n &:= \frac{3336726053}{200000000000}, & d &:= \frac{151371587707}{1000000000000}. \end{aligned}$$

The cleared degree-nine polynomial is strictly positive on $[0, C^2]$; the exact certificate uses 21 terminal intervals. Equation (204) then gives

$$g_v(4373/2000) \leq A - Bz_v^4, \quad A := \frac{1382567389148960648189}{100000000000000000000}, \quad B := \frac{5546772823}{31250000000}. \quad (211)$$

The identity defining A in Eq. (204) is exact.

One further schedule Set

$$m_0 := \frac{5001868011}{2500000000}, \quad s_0 := \frac{1143807}{1250000}.$$

Then $0 < s_0 < 1$, $m_0 > Cs_0$, and $m_0/s_0 = 4373/2000$. Use A, B from Eq. (211) and put

$$U_z := \frac{A - Bz^4}{s_0}, \quad x := \frac{s - s_0}{1 - s_0}, \quad k(s) := 10^{-12} \sum_{i=0}^9 a_i x^i.$$

Set $a_i := [x^i]P^{\text{fin}}(x)$, where

$$\begin{aligned}
P^{\text{fin}}(x) := & 1053943458776 + 34491607828x - 760392537x^2 \\
& - 35161907x^3 + 2241981x^4 - 92292x^5 \\
& + 13387x^6 - 698x^7 - 60x^8 \\
& + 9x^9.
\end{aligned} \tag{212}$$

Its exact antiderivative is

$$m(s) := m_0 + 2(1 - s_0)10^{-12} \sum_{i=0}^9 a_i \left(\frac{s_0 x^{i+1}}{i+1} + \frac{(1-s_0)x^{i+2}}{i+2} \right).$$

Thus $m'(s) = 2sk(s)$ and $m(s_0) = m_0$. On $s_0 \leq s \leq 1$, the exact certificate proves

$$k(s) > \frac{A}{2s_0}, \quad m(s) > \frac{15}{8}s,$$

and, with the compression polynomials evaluated at $(m(s), s, z)$,

$$\begin{aligned} D_0 + (k(s) - U_z)J_0 &> 0 & (0 \leq z \leq 1/2), \\ D_1 + (k(s) - U_z)J_1 &> 0 & (1/2 \leq z \leq 1). \end{aligned}$$

After the substitutions $s = s_0 + (1 - s_0)x$ and either $z = y/2$ or $z = (1 + y)/2$, these four polynomials have bidegrees $(9, 0)$, $(11, 0)$, $(32, 7)$, and $(44, 9)$. The first three have positive Bernstein coefficients on the entire unit square. The fourth is certified on 165 terminal rectangles, of maximum subdivision depth twelve. All coefficients are strictly positive. Compactness supplies the uniform margins required by Lemma 4.43, including the partial-round error at $z = 0$. That lemma makes

$$E := m(1) = \frac{9420549260968121520502759549}{433125000000000000000000000000} \quad (213)$$

admissible. In particular,

$$\frac{108751}{50000} - E = \frac{6114031878479497240451}{4331250000000000000000000} > 0.$$

Consequently $E = 2.17501858839090828756 \dots < 2.17502$. The original-support argument below now applies with $c_* = 108751/50000$.

Exact data and verification The six new constraints and the final schedule require 76494 positive terminal Bernstein coefficient checks. Together with the 76176 checks in the preceding resolvent certificate, this gives 152670 checks for the 26 resolvent constraints and three schedules. The input 2.19595 is established in Section 4.11.

4.14 Original supports and Lyapunov approximation

We now transfer the admissible polynomial bound to choices in the original supports and derive the subset-approximation consequence.

If $\sigma = 0$, every nonzero summand has zero variance and the discrepancy vanishes for its support choices. Otherwise set $B_i := \tau_i A_i / \sigma$. For independent centered variance-one variables η_i , rank one makes each determinant affine in every η_i . Expanding the product shows

$$\mathbb{E}[\det(x^2 I - (\sum_{i=1}^n \eta_i B_i)^2)] = P_B(x).$$

Only the zeroth and second moments survive, so this holds for the standardized original variables, with deterministic ones omitted.

For completeness, we verify real-rootedness of every convex average required by interlacing, keeping the original means μ_i fixed in the discrepancy. Omit deterministic variables, so $\tau_i > 0$ for the remaining indices, and relabel them. At a node of the support tree, fix the previously selected outcomes. An arbitrary convex combination of the children replaces the distribution of the next variable by a probability distribution on its original support. The remaining variables are still independent. Denote their current means by m'_i and variances by $v'_i \geq 0$. Absorb the fixed outcomes and all mean shifts $(m'_i - \mu_i)A_i$ into a fixed Hermitian offset H_0 . For the r remaining variables the starting polynomial is

$$Q(x, z) := \det(xI + H_0 + \sum_{i=1}^r z_i \tau_i A_i) \det(xI - H_0 + \sum_{i=1}^r z_i \tau_i A_i).$$

It is real stable because both matrix factors have positive definite imaginary part when all variables are in the upper half-plane. Rank one makes each determinant affine in each coefficient. Centering the current distributions at m'_i and expanding the product therefore gives their expected characteristic polynomial as

$$[\prod_{i=1}^r (1 - \frac{v'_i}{2\tau_i^2} \partial_{z_i}^2) Q(x, z)]_{z=0}.$$

The variance-dependent operator factors as

$$1 - \frac{v'_i}{2\tau_i^2} \partial_{z_i}^2 = (1 - \frac{\sqrt{v'_i}}{\sqrt{2}\tau_i} \partial_{z_i})(1 + \frac{\sqrt{v'_i}}{\sqrt{2}\tau_i} \partial_{z_i}).$$

Thus it preserves real stability, including the identity operator when $v'_i = 0$. Real specialization yields a real-rooted polynomial; its leading coefficient in x is one, so the zero-polynomial exception cannot occur.

Use conditional expected polynomials at the nodes and unweighted characteristic polynomials at the leaves. Every leaf polynomial, and hence every conditional expectation, is monic of the same degree $2d$. The preceding argument applies to every convex combination of the children, so the common-interlacing criterion applies at every node. All original supports are finite, making this a finite tree. Repeated selection therefore gives a leaf whose largest root is at most that of the initial expectation. At a leaf this root is exactly the norm of the Hermitian sum centered at the original means μ_i . The initial expectation is $\sigma^{2d} P_B(x/\sigma)$, so admissibility of c_* proves Theorem 4.1 for the original supports. This is the selection principle used in [KLS20].

Bernoulli coefficients turn the discrepancy bound into an approximation of fractional matrix sums by subset sums.

Corollary 4.45. *If $\sum_{i=1}^n u_i u_i^* \preceq I$ and $\|u_i\|^2 \leq \epsilon$, then for every $t_i \in [0, 1]$ there is a subset S such that*

$$\left\| \sum_{\substack{i=1 \\ i \in S}}^n u_i u_i^* - \sum_{i=1}^n t_i u_i u_i^* \right\| \leq \frac{c_*}{2} \sqrt{\epsilon} \leq 1.08751 \sqrt{\epsilon}.$$

Proof. For independent Bernoulli variables of means t_i , $\sum_{i=1}^n t_i(1 - t_i)A_i^2 \preceq (\epsilon/4)I$. Apply Theorem 4.1 and take S to be the indices whose selected Bernoulli outcome equals one. $\square$

The bound holds without an isotropy, dimension, or effective-rank restriction. The exact Bernstein coefficient checks, including the 76176 checks in Section 4.12 and the 76494 additional checks in Section 4.13, certify the schedules over their full intervals or rectangles; the dimension-independent deduction uses the compression, moment, tensor, barrier, and interlacing proofs above.

5 A lower bound of two

We give a probabilistic construction showing that every universal rank-one discrepancy constant is at least 2. The examples use real vectors and symmetric Rademacher coefficients. The construction is asymptotic: for each $c < 2$ it yields finite examples with discrepancy ratio greater than c , without claiming that a finite example attains exactly 2 or that 2 is a universal upper bound.

5.1 Statement and normalization

For real vectors $u_1, \dots, u_n \in \mathbb{R}^d$, put

$$A_i := u_i u_i^\top, \quad \text{Disc}(A) := \min_{\varepsilon \in \{-1, 1\}^n} \left\| \sum_{i=1}^n \varepsilon_i A_i \right\|, \quad \sigma(A)^2 := \left\| \sum_{i=1}^n A_i^2 \right\|.$$

All matrix norms without subscripts are operator norms. Thus $\sigma(A)$ is exactly the variance normalization for independent symmetric Rademacher coefficients. In particular these examples are allowed by the original-support formulation with arbitrary finite real supports.

The lower bound follows from Gaussian families with sufficiently many vectors relative to the dimension.

Theorem 5.1. *For every $0 < c < 2$, there exist finite d, n and real vectors u_i such that*

$$\text{Disc}(A) > c\sigma(A).$$

Consequently the optimal universal rank-one discrepancy constant is at least 2, even when all coefficient supports are $\{-1, 1\}$ and all vectors are real.

The earlier unit-norm tight-frame construction with $n = 2d - 1$ gives the ratio $\sqrt{2 - 1/d}$, tending to $\sqrt{2}$ [XXZ21]. The construction below instead takes $n := \alpha d$, where α is a sufficiently large fixed integer, and then lets d grow. Gaussian Wishart examples have also been used in average-case matrix discrepancy [KOWZ25]; their Theorem 3 states bounds up to unspecified constants. Pesenti also uses Gaussian vectors and a spectral union bound for a lower bound on unweighted sparsification [Pes26]. These related results concern different stated bounds or objectives. The uniform, variance-normalized claim here is proved directly below.

5.2 Elementary probabilistic ingredients

Gaussian concentration will control the signed trace moments, while chi-square estimates will bound the variance parameter.

Lemma 5.2. *If X is a standard real Gaussian vector and f is L -Lipschitz for Euclidean distance, then*

$$\Pr[f(X) \leq \mathbb{E}[f(X)] - t] \leq \exp\left(-\frac{t^2}{2L^2}\right) \quad (t > 0).$$

If Z has the chi-square distribution with q degrees of freedom, then for $0 < t < 1$,

$$\Pr[|Z/q - 1| > t] \leq 2e^{-qt^2/8}, \quad \mathbb{E}[Z^r] = \prod_{j=0}^{r-1} (q + 2j) \quad (r \in \mathbb{N}).$$

Proof. For completeness, Gaussian concentration follows from the Gaussian logarithmic Sobolev inequality

$$\text{Ent}_\gamma(h) \leq \frac{1}{2} \int \frac{\|\nabla h\|^2}{h} d\gamma.$$

One derivation uses the Ornstein–Uhlenbeck semigroup $P_s h(x) := \mathbb{E}[h(e^{-s}x + \sqrt{1 - e^{-2s}}X)]$. Gaussian integration by parts, invariance of γ , and convergence of $P_s h$ to $\int h d\gamma$ give

$$\text{Ent}_\gamma(h) = \int_0^\infty \int \frac{\|\nabla P_s h\|^2}{P_s h} d\gamma ds.$$

The identity $\nabla P_s h = e^{-s} P_s \nabla h$ and weighted Cauchy–Schwarz bound the integrand by

$$e^{-2s} \int \|\nabla h\|^2 / h d\gamma.$$

Integration gives the claimed inequality, first for smooth bounded positive functions, and then by approximation. Apply it to $h := e^{\lambda f}$. For $\psi(\lambda) := \log \mathbb{E}[e^{\lambda f}]$ this says $\lambda \psi'(\lambda) - \psi(\lambda) \leq \lambda^2 L^2 / 2$. Integration and the limit at zero give $\mathbb{E}[e^{\lambda(f - \mathbb{E}[f])}] \leq e^{\lambda^2 L^2 / 2}$. Chernoff's inequality applied to $-f$ proves the concentration statement. Smoothing and truncation cover all Lipschitz functions.

For the remaining statements, use $\mathbb{E}[e^{sZ}] = (1 - 2s)^{-q/2}$ for $s < 1/2$. Optimizing the Chernoff bound yields exponents $-q(t - \log(1 + t))/2$ for the upper tail and $-q(-t - \log(1 - t))/2$ for the lower tail. Both are at most $-qt^2/8$ for $0 < t < 1$. Differentiation of the same moment generating function gives the moments. $\square$

The next estimates control the covariance and column norms of the Gaussian family, hence its variance parameter.

Lemma 5.3. *Fix $0 < \delta < 1$ and an integer α such that $b := \alpha\delta^2/32 - \log 9 > 0$. Let $n := \alpha d$ and let $G := [g_1 \ \cdots \ g_n] \in \mathbb{R}^{d \times n}$ have independent $N(0, 1)$ entries. Then*

$$\Pr\left[\left\|\frac{1}{n}GG^\top - I\right\| > \delta\right] \leq q_d := 2e^{-bd}, \quad (214)$$

$$\Pr\left[\max_{i \in [n]} \frac{\|g_i\|^2}{d} > 1 + \delta\right] \leq 2\alpha d e^{-d\delta^2/8}. \quad (215)$$

In particular, for $u_i := g_i/\sqrt{d}$, with probability tending to one,

$$\sigma(A)^2 \leq \alpha(1 + \delta)^2. \quad (216)$$

Proof. A maximal $1/4$ -separated subset $\mathcal{N}$ of the real unit sphere is a $1/4$ -net of size at most 9^d , by disjoint-ball volume comparison. For any symmetric B , $\|B\| \leq 2 \max_{x \in \mathcal{N}} |x^\top B x|$: approximate a unit eigenvector attaining $\|B\|$, and bound the two error terms by $\|B\|/4$ each. For each fixed unit vector x , $\|G^\top x\|^2$ is chi-square with n degrees of freedom. Lemma 5.2, with $t = \delta/2$, and a union bound over the net prove Eq. (214). The same lemma with $q = d, t = \delta$ proves Eq. (215). Finally,

$$\sum_{i=1}^n A_i^2 = \sum_{i=1}^n \|u_i\|^2 u_i u_i^\top \preceq (\max_{i \in [n]} \|u_i\|^2) \frac{G G^\top}{d} \preceq \alpha(1 + \delta)^2 I$$

on the intersection of the two events. $\square$

5.3 A moment lower bound for every signing

Write $\text{Cat}_k := \frac{1}{k+1} \binom{2k}{k}$ for the Catalan number.

Even trace moments provide a spectral lower bound with the same expectation estimate for every signing.

Lemma 5.4. *Let d, n, k be positive integers, let $G \in \mathbb{R}^{d \times n}$ have independent $N(0, 1)$ entries, and let $\varepsilon \in \{-1, 1\}^n$ be deterministic. Set*

$$S_\varepsilon := \frac{1}{d} G \text{diag}(\varepsilon) G^\top.$$

Then

$$\mathbb{E}\left[\frac{1}{d} \text{tr}[S_\varepsilon^{2k}]\right] \geq \text{Cat}_k(n/d)^k. \quad (217)$$

Proof. Set $m := 2k$ and expand the trace as

$$d^{-m-1} \sum_{a_1, \dots, a_m=1}^d \sum_{i_1, \dots, i_m=1}^n \prod_{\ell=1}^m \varepsilon_{i_\ell} G_{a_\ell i_\ell} G_{a_{\ell+1} i_\ell}, \quad a_{m+1} := a_1. \quad (218)$$

The real Gaussian Wick formula expands the expectation of the $2m$ entries into a sum over pairings, with each pair imposing equality of the two row indices and of the two column indices. The formula follows directly by differentiating the Gaussian moment generating function $\mathbb{E}[e^{\langle t, X \rangle}] = e^{\|t\|^2/2}$.

Crucially, the contribution of *each entire pairing after summing indices* is nonnegative. Indeed, its row constraints contribute d^r for some r . Let $\mathcal{B}$ be the partition of $[m]$ into classes imposed by its column constraints; their contribution is

$$\prod_{B \in \mathcal{B}} \sum_{i=1}^n \varepsilon_i^{|B|}.$$

Each even-sized class contributes n and each odd-sized class contributes $s := \sum_{i=1}^n \varepsilon_i$. The number of odd-sized classes is even because m is even. Thus the product is nonnegative, even when $s < 0$. Distinct classes are summed independently; their assigned column indices need not be distinct. This observation includes unbalanced signings.

We may therefore retain only selected pairings. Label the two factors at position ℓ in Eq. (218) by L_ℓ and R_ℓ . For each noncrossing pairing π of $[2k]$, pair L_ℓ with $R_{\pi(\ell)}$ for all ℓ . These are valid, distinct Wick pairings of the $4k$ factors. Their column classes are the k pairs of π , giving n^k . Their row constraints are $a_\ell = a_{\pi(\ell)+1}$, so they have $k+1$ free row classes. Here indices are cyclic. The identity of $k+1$ follows by removing an adjacent pair in the noncrossing pairing and inducting,

starting with two positions and two free rows. There are Cat_k such noncrossing pairings: pairing the first position with position $2j$ gives the recurrence $\text{Cat}_k = \sum_{j=1}^k \text{Cat}_{j-1} \text{Cat}_{k-j}$, $\text{Cat}_0 = 1$. Each retained contribution, including the normalization, is $d^{-2k-1} d^{k+1} n^k = (n/d)^k$. This proves Eq. (217). $\square$

5.4 Making the estimate simultaneous over all signings

Gaussian concentration upgrades the expectation estimate to a simultaneous bound over all signings.

Lemma 5.5. *Fix $k \geq 1$, $0 < \delta < 1$, and an integer α satisfying the condition of Lemma 5.3. Set $n := \alpha d$ and let $G \in \mathbb{R}^{d \times n}$ have independent $N(0, 1)$ entries. Then, with probability tending to one as $d \rightarrow \infty$,*

$$\frac{1}{d} \text{tr}[S_\varepsilon^{2k}] \geq \frac{1}{2} \text{Cat}_k \alpha^k \quad \text{for every } \varepsilon \in \{-1, 1\}^n. \quad (219)$$

Proof. Put $R := \sqrt{\alpha(1+\delta)}$ and introduce the *common* event

$$K := \{G : \|G\| \leq R\sqrt{d}\}.$$

Lemma 5.3 gives $\Pr[K^c] \leq q_d$. For every $G \in \mathbb{R}^{d \times n}$, define $F_\varepsilon(G) := d^{-1} \text{tr}[S_\varepsilon(G)^{2k}]$. On K , this function takes values in $[0, R^{4k}]$ and is Lipschitz for Frobenius distance with constant

$$L_d := \frac{4kR^{4k-1}}{d}. \quad (220)$$

In fact, for $G, H \in K$,

$$\|S_\varepsilon(G) - S_\varepsilon(H)\|_F \leq \frac{2R}{\sqrt{d}} \|G - H\|_F.$$

The telescoping identity for $X^{2k} - Y^{2k}$, cyclicity of trace, and $|\text{tr}[UV]| \leq \|U\|_F \|V\|_F$ show that

$$|d^{-1} \text{tr}[X^{2k} - Y^{2k}]| \leq \frac{2k}{\sqrt{d}} R^{4k-2} \|X - Y\|_F$$

when $\|X\|, \|Y\| \leq R^2$, proving Eq. (220).

Extend the restriction $F_\varepsilon|_K$ to all real $d \times n$ matrices by

$$\widehat{F}_\varepsilon(G) := \min\{R^{4k}, \max\{0, \inf_{H \in K} (F_\varepsilon(H) + L_d \|G - H\|_F)\}\}.$$

This extension agrees with F_ε on K , is L_d -Lipschitz, and takes values in $[0, R^{4k}]$. The standard infimum extension properties follow immediately from the triangle inequality and the Lipschitz property on K .

There is a bound independent of the signing,

$$|\mathbb{E}[\widehat{F}_\varepsilon] - \mathbb{E}[F_\varepsilon]| \leq e_d, \quad (221)$$

$$e_d := d^{-2k} \left[\prod_{j=0}^{4k-1} (dn + 2j) \right]^{1/2} q_d^{1/2} + R^{4k} q_d = o(1).$$

Indeed, globally $0 \leq F_\varepsilon(G) \leq (\|G\|_F^2/d)^{2k}$. Apply Cauchy–Schwarz on K^c and the chi-square moment formula with dn degrees of freedom. For fixed k, α, δ , the first factor in Eq. (221) grows only polynomially in d , whereas q_d decays exponentially.

For sufficiently large d , Lemma 5.4 now gives $\mathbb{E}[\widehat{F}_\varepsilon] \geq \frac{3}{4}\text{Cat}_k\alpha^k$, uniformly in ε . Lemma 5.2 with $t = \text{Cat}_k\alpha^k/4$ yields

$$\Pr[\widehat{F}_\varepsilon < \frac{1}{2}\text{Cat}_k\alpha^k] \leq e^{-ad^2}, \quad a := \frac{\text{Cat}_k^2\alpha^{2k}}{512k^2R^{8k-2}} > 0.$$

Taking a union bound over all 2^n signings and then excluding K^c shows that the failure probability in Eq. (219) is at most

$$q_d + \exp(\alpha d \log 2 - ad^2) = o(1). \quad (222)$$

The exceptional event K^c is common to all signings and is counted *once*, not 2^n times. All parameters except d were fixed before taking the limit. $\square$

5.5 Completion and quantitative interpretation

Proof of Theorem 5.1. Since $d^{-1} \text{tr}[S^{2k}] \leq \|S\|^{2k}$ for symmetric S , Lemma 5.5 implies

$$\text{Disc}(A) \geq (\text{Cat}_k/2)^{1/(2k)}\sqrt{\alpha}.$$

Combining this with Lemma 5.3 gives, with probability tending to one,

$$\frac{\text{Disc}(A)}{\sigma(A)} \geq \frac{(\text{Cat}_k/2)^{1/(2k)}}{1 + \delta}. \quad (223)$$

Finally $(\text{Cat}_k/2)^{1/(2k)} \rightarrow 2$. For example, $4^k/((k+1)(2k+1)) \leq \text{Cat}_k \leq 4^k$ proves this directly. Given $c < 2$, first choose k , then $\delta > 0$ small enough so the right side of Eq. (223) exceeds c , then fix an integer $\alpha > 32 \log 9/\delta^2$, and finally choose sufficiently large d . The intersection of the required events has positive probability, so at least one finite deterministic realization has the claimed property. Gaussian columns are nonzero almost surely, hence its matrices have rank exactly one. $\square$

One finite parameter recipe follows directly from the proof. Choose k, δ, α, d such that $b > 0$,

$$e_d \leq \frac{1}{4}\text{Cat}_k\alpha^k, \quad 2q_d + e^{\alpha d \log 2 - ad^2} + 2\alpha d e^{-d\delta^2/8} < 1.$$

Then a successful realization exists and obeys Eq. (223). These bounds are deliberately conservative; the argument does not offer a practical small-dimensional witness.

For a concrete strict improvement over $\sqrt{2}$, take $k := 10$ and $\delta := 1/100$. The integer inequality

$$\frac{\text{Cat}_{10}}{2} = 8398 > \left(\frac{303}{200}\right)^{20}$$

implies that Eq. (223) exceeds $3/2$. For example $\alpha := 2,000,000$ meets $b > 0$; sufficiently large d completes this finite existence example. Taking larger k and smaller δ makes the ratio exceed any prescribed number below 2.

The conclusion $C_0 \geq 2$ is a statement about the supremum over finite instances. It does not assert that one finite instance attains exactly 2, nor that 2 is a universal upper bound. The construction uses the same real rank-one matrices and the same σ as the discrepancy problem; there is no passage to higher-rank matrices or enlarged coefficient supports.

6 Lower bounds in the square-root normalization

The variance-normalized lower bound does not by itself imply the same coefficient under the isotropic hypotheses of the Kadison–Singer signing problem. We obtain that conclusion by normalizing the Gaussian construction of Section 5 to have covariance exactly equal to the identity. The resulting loss tends to one.

6.1 Statement and normalization

For a finite real family $v_1, \dots, v_n \in \mathbb{R}^d$, write

$$B_i := v_i v_i^\top, \quad D(v) := \min_{s \in \{-1, 1\}^n} \left\| \sum_{i=1}^n s_i B_i \right\|.$$

Here $D(v)$ is the discrepancy for independent symmetric signs.

The next theorem transfers the Gaussian lower bound to the $\sqrt{\epsilon}$ scale while imposing exact isotropy and a prescribed maximum squared vector norm.

Theorem 6.1. *For every $0 < c < 2$, there is $\epsilon_0 > 0$ such that, for every $0 < \epsilon \leq \epsilon_0$, there are finite d, n and real vectors $v_1, \dots, v_n \in \mathbb{R}^d$ satisfying*

$$\sum_{i=1}^n v_i v_i^\top = I_d, \quad \max_{i \in [n]} \|v_i\|^2 = \epsilon, \quad (224)$$

for which

$$\left\| \sum_{i=1}^n s_i v_i v_i^\top \right\| > c\sqrt{\epsilon} \quad \text{for every } s \in \{-1, 1\}^n. \quad (225)$$

Consequently, the $\sqrt{\epsilon}$ order cannot be replaced by $o(\sqrt{\epsilon})$, and any uniform asymptotic leading coefficient in the signing bound is at least 2.

The assertion concerns the worst case over finite families in arbitrary dimension. It does not assert that every isotropic family has large discrepancy, that a finite family attains the coefficient 2, or that $2\sqrt{\epsilon}$ is a universal upper bound.

6.2 Normalization by the covariance

The next lemma quantifies how covariance normalization changes the vector norms and discrepancy.

Lemma 6.2. *Let $a_1, \dots, a_N \in \mathbb{R}^d$ and $H := \sum_{i=1}^N a_i a_i^\top$. Suppose that $h_- I_d \preceq H \preceq h_+ I_d$ for some $h_+, h_- > 0$ and that $\max_{i \in [N]} \|a_i\|^2 \leq M$. Set $v_i := H^{-1/2} a_i$. Then*

$$\sum_{i=1}^N v_i v_i^\top = I_d, \quad \max_{i \in [N]} \|v_i\|^2 \leq \frac{M}{h_-}, \quad D(v) \geq \frac{D(a)}{h_+}. \quad (226)$$

Proof. The covariance identity follows by congruence. Also $\|v_i\|^2 = a_i^\top H^{-1} a_i \leq \|a_i\|^2 / h_-$. For each signing s , let $T_s := \sum_{i=1}^N s_i v_i v_i^\top$. The identity

$$\sum_{i=1}^N s_i a_i a_i^\top = H^{1/2} T_s H^{1/2}$$

and submultiplicativity of the operator norm give $\left\| \sum_{i=1}^N s_i a_i a_i^\top \right\| \leq h_+ \|T_s\|$. Taking minima proves the last inequality in Eq. (226). This argument applies to indefinite signed sums. $\square$

Proof of Theorem 6.1. Let $\text{Cat}_k := \frac{1}{k+1} \binom{2k}{k}$ and $m_k := (\text{Cat}_k/2)^{1/(2k)}$. As shown in Section 5, $m_k \rightarrow 2$. First choose $k \geq 1$ and then $0 < \delta < 1$ so that

$$\frac{m_k \sqrt{1-\delta}}{(1+\delta)^{3/2}} > c. \quad (227)$$

Put $\rho := (1+\delta)/(1-\delta)$ and $a_{\text{low}} := m_k/(1+\delta)$. By Eq. (227), $a_{\text{low}}/\sqrt{\rho} > c$. Choose $0 < \epsilon_0 < 1$ sufficiently small that

$$\frac{a_{\text{low}}}{\sqrt{\rho + \epsilon_0}} > c, \quad \frac{\rho}{\epsilon_0} > \frac{32 \log 9}{\delta^2}. \quad (228)$$

Fix any $0 < \epsilon \leq \epsilon_0$ and set $\alpha := \lceil \rho/\epsilon \rceil$. This is an integer satisfying $\alpha\delta^2/32 - \log 9 > 0$. Keep $k, \delta, \epsilon, \alpha$ fixed while letting d grow, and put $N := \alpha d$. Let $G := [g_1 \cdots g_N] \in \mathbb{R}^{d \times N}$ have independent standard real Gaussian entries, and set $a_i := g_i/\sqrt{d}$. By Lemma 5.3, with probability tending to one,

$$\alpha(1-\delta)I_d \preceq H := \frac{1}{d}GG^\top \preceq \alpha(1+\delta)I_d, \quad \max_{i \in [N]} \|a_i\|^2 \leq 1 + \delta. \quad (229)$$

The lower covariance inequality here uses the two-sided event in Eq. (214). By Lemma 5.5 and $d^{-1} \text{tr}[S^{2k}] \leq \|S\|^{2k}$ for real symmetric S , also with probability tending to one,

$$D(a) \geq m_k \sqrt{\alpha}. \quad (230)$$

The intersection of these events has positive probability for all sufficiently large d . Fix one such realization with nonzero columns.

Apply Lemma 6.2 with $h_- := \alpha(1-\delta)$, $h_+ := \alpha(1+\delta)$, and $M := 1 + \delta$. The vectors $v_i := H^{-1/2}a_i$ satisfy

$$\sum_{i=1}^N v_i v_i^\top = I_d, \quad \max_{i \in [N]} \|v_i\|^2 \leq \frac{\rho}{\alpha} \leq \epsilon, \quad D(v) \geq \frac{a_{\text{low}}}{\sqrt{\alpha}}. \quad (231)$$

The ceiling choice gives $\alpha\epsilon \leq \rho + \epsilon$. Consequently, Eq. (228) implies

$$D(v) \geq \frac{a_{\text{low}} \sqrt{\epsilon}}{\sqrt{\alpha\epsilon}} \geq \frac{a_{\text{low}} \sqrt{\epsilon}}{\sqrt{\rho + \epsilon_0}} > c\sqrt{\epsilon}. \quad (232)$$

Thus the desired bound holds for every sufficiently small prescribed ϵ , rather than only along a selected sequence.

To make the maximum squared norm exactly ϵ , if necessary adjoin one orthogonal coordinate. Let $q := \lceil 1/\epsilon \rceil$, $r_j := \epsilon$ for $j \in [q-1]$, and $r_q := 1 - (q-1)\epsilon$. Then $0 < r_q \leq \epsilon$ and $\sum_{j=1}^q r_j = 1$. Extend the previous vectors by a zero last coordinate and append $\sqrt{r_j}e_{d+1}$, $j \in [q]$. The enlarged covariance is I_{d+1} , and its maximum squared vector norm is exactly ϵ since $q \geq 2$. Every signed sum is block diagonal, with the previous signed sum as its first block, so Eq. (232) remains valid. Relabeling this finite family proves the theorem. $\square$

6.3 Partition and Lyapunov consequences

The signing lower bound also gives obstructions to balanced partitions and to approximating $I_d/2$ by a subset sum.

Corollary 6.3. *For every $0 < c < 2$ and every sufficiently small $\epsilon > 0$, there is a family satisfying Eq. (224) such that every subset $S \subseteq [n]$ satisfies*

$$\left\| \sum_{\substack{i=1 \\ i \in S}}^n v_i v_i^\top - \frac{1}{2} I_d \right\| > \frac{c}{2} \sqrt{\epsilon}. \quad (233)$$

Moreover, for every partition $[n] = S \dot{\cup} S^c$,

$$\max\left\{ \left\| \sum_{\substack{i=1 \\ i \in S}}^n v_i v_i^\top \right\|, \left\| \sum_{\substack{i=1 \\ i \in S^c}}^n v_i v_i^\top \right\| \right\} > \frac{1}{2} + \frac{c}{2} \sqrt{\epsilon}. \quad (234)$$

In particular, the asymptotic leading coefficient in a uniform Lyapunov subset-approximation bound is at least 1, already for the target weights $t_i := 1/2$.

Proof. Set $B_i := v_i v_i^\top$ and assign $s_i := 1$ on S and $s_i := -1$ on S^c . Isotropy gives

$$T := \sum_{i=1}^n s_i B_i = 2 \sum_{\substack{i=1 \\ i \in S}}^n B_i - I_d.$$

Theorem 6.1 proves Eq. (233). Both subset sums are positive semidefinite and equal to $(I_d + T)/2$ and $(I_d - T)/2$. Their larger operator norm is $1/2 + \|T\|/2$, proving Eq. (234). $\square$

For comparison with the variance normalization, the isotropic family also satisfies

$$\sum_{i=1}^n B_i^2 = \sum_{i=1}^n \|v_i\|^2 B_i \preceq \epsilon I_d.$$

This inequality explains the upper-bound specialization $\sigma \leq \sqrt{\epsilon}$. The lower bound above required the separate covariance-normalization argument; it does not follow from that one-sided variance inequality alone.

6.4 An elementary explicit example for the exponent

The necessity of the square-root order also has a short deterministic proof, with a smaller constant, independent of the Gaussian lemmas.

Proposition 6.4. *For every integer $m \geq 2$, there is an isotropic real family of $\binom{m}{2}$ vectors in dimension $m - 1$, all of squared norm $\epsilon_m := 2/m$, such that every signing has operator norm at least $\sqrt{\epsilon_m/2}$.*

Proof. Work first in $\mathcal{H} := \{x \in \mathbb{R}^m : \sum_{a=1}^m x_a = 0\}$ and, for each pair $1 \leq a < b \leq m$, set $w_{ab} := (e_a - e_b)/\sqrt{m}$. Writing $\mathbf{1} := (1, \dots, 1)^\top$, direct entrywise calculation gives

$$\sum_{a=1}^{m-1} \sum_{b=a+1}^m w_{ab} w_{ab}^\top = I_m - \frac{1}{m} \mathbf{1} \mathbf{1}^\top.$$

This is the identity on $\mathcal{H}$, and $\|w_{ab}\|^2 = 2/m$. For any signs $s_{ab} \in \{-1, 1\}$, put

$$T := \sum_{a=1}^{m-1} \sum_{b=a+1}^m s_{ab} w_{ab} w_{ab}^\top.$$

Its off-diagonal entries have absolute value $1/m$, so

$$\mathrm{tr}[T^2] = \sum_{a=1}^m \sum_{b=1}^m T_{ab}^2 \geq \frac{m(m-1)}{m^2} = \frac{m-1}{m}.$$

Also $T\mathbf{1} = 0$, hence at most $m-1$ eigenvalues are nonzero. It follows that $\|T|_{\mathcal{H}}\|^2 \geq \mathrm{tr}[T^2]/(m-1) \geq 1/m$. Choose an orthonormal basis of $\mathcal{H}$ to identify these vectors with vectors in $\mathbb{R}^{m-1}$. This preserves their norms, covariance, and signed operator norms, and proves the proposition. $\square$

AI Usage Disclosure

We used Claude Code Fable 5, ChatGPT Pro 5.6, and Codex Sol 5.6 in preparing this draft. We used these tools to generate major proof ideas of the upper bound. The hard instance for lower bound is suggested by human authors. And AI comes up with the calculations and proofs for lower bounds. All of the proofs have been substantially rewritten by the authors. All of the proofs have been verified by different AI models and also human authors as well. The authors take full responsibilities for the correctness of the proof.

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A Exact data for the discrepancy bound

We record the polynomial constructions and Gram-block specifications used in Section 4. The certificate conditions and admissibility arguments are given with the corresponding proof stages.

A.1 Eighth-moment schedule polynomials

For stage $j = 1, \dots, 6$ in Eq. (131), define $a_i := [x^i]P_j^{(8)}(x)$ by

$$\begin{aligned} P_1^{(8)}(x) := & 1029906760149 + 107231987546x - 8398860358x^2 \\ & - 915341417x^3 + 345772738x^4 - 58626360x^5 \\ & + 9973094x^6 - 1880836x^7 + 285850x^8 \\ & - 25269x^9. \end{aligned} \tag{235}$$

$$\begin{aligned} P_2^{(8)}(x) := & 1034514865274 + 97684240277x - 7064950996x^2 \\ & - 694997060x^3 + 236247589x^4 - 34153841x^5 \\ & + 4864149x^6 - 795083x^7 + 88959x^8 \\ & - 2083x^9. \end{aligned} \tag{236}$$

$$\begin{aligned} P_3^{(8)}(x) := & 1036013498344 + 94910100064x - 6698665434x^2 \\ & - 638099703x^3 + 210230444x^4 - 28941351x^5 \\ & + 3871531x^6 - 586404x^7 + 48584x^8 \\ & + 3087x^9. \end{aligned} \tag{237}$$

$$\begin{aligned}
P_4^{(8)}(x) := & 1036638980263 + 94115107212x - 6594916843x^2 \\
& - 622390475x^3 + 203148761x^4 - 27553070x^5 \\
& + 3618712x^6 - 539921x^7 + 42360x^8 \\
& + 3445x^9.
\end{aligned} \tag{238}$$

$$\begin{aligned}
P_5^{(8)}(x) := & 1036709375298 + 93882557368x - 6565163993x^2 \\
& - 617775023x^3 + 201141951x^4 - 27177247x^5 \\
& + 3553892x^6 - 529061x^7 + 41443x^8 \\
& + 3368x^9.
\end{aligned} \tag{239}$$

$$\begin{aligned}
P_6^{(8)}(x) := & 1036724690384 + 93787146104x - 6552851583x^2 \\
& - 615961224x^3 + 200339762x^4 - 27026524x^5 \\
& + 3527981x^6 - 524914x^7 + 41209x^8 \\
& + 3305x^9.
\end{aligned} \tag{240}$$

A.2 Tenth-moment schedule polynomials

For stage $j = 1, \dots, 7$ in Section 4.8, define $c_i := [x^i]P_j^{(10)}(x)$ by

$$\begin{aligned}
P_1^{(10)}(x) := & 1039793718042 + 83680489893x - 6323673299x^2 \\
& - 140568929x^3 + 97723312x^4 - 19752136x^5 \\
& + 4097062x^6 - 729897x^7 + 88339x^8 \\
& - 5514x^9.
\end{aligned} \tag{241}$$

$$\begin{aligned}
P_2^{(10)}(x) := & 1042498691496 + 78037618953x - 5539774761x^2 \\
& - 115023501x^3 + 73243474x^4 - 13273790x^5 \\
& + 2536170x^6 - 427078x^7 + 48819x^8 \\
& - 2845x^9.
\end{aligned} \tag{242}$$

$$\begin{aligned}
P_3^{(10)}(x) := & 1043509138300 + 76056476227x - 5276794547x^2 \\
& - 106695773x^3 + 65906719x^4 - 11478083x^5 \\
& + 2126987x^6 - 350936x^7 + 39333x^8 \\
& - 2244x^9.
\end{aligned} \tag{243}$$

$$\begin{aligned}
P_4^{(10)}(x) := & 1043874593852 + 75354033942x - 5185065183x^2 \\
& - 103825133x^3 + 63449119x^4 - 10892982x^5 \\
& + 1996074x^6 - 326731x^7 + 36273x^8 \\
& - 2036x^9.
\end{aligned} \tag{244}$$

$$\begin{aligned}
P_5^{(10)}(x) := & 1043983647889 + 75106728213x - 5152982955x^2 \\
& - 102823762x^3 + 62603826x^4 - 10694502x^5 \\
& + 1952283x^6 - 318901x^7 + 35395x^8 \\
& - 1998x^9.
\end{aligned} \tag{245}$$

$$\begin{aligned}
P_6^{(10)}(x) := & 1044051346040 + 75020890334x - 5141834890x^2 \\
& - 102479127x^3 + 62308926x^4 - 10624563x^5 \\
& + 1936732x^6 - 316039x^7 + 35031x^8 \\
& - 1972x^9.
\end{aligned} \tag{246}$$

$$\begin{aligned}
P_7^{(10)}(x) := & 1044019914969 + 74990784283x - 5137993167x^2 \\
& - 102354319x^3 + 62212035x^4 - 10603207x^5 \\
& + 1932091x^6 - 315184x^7 + 34927x^8 \\
& - 1966x^9.
\end{aligned} \tag{247}$$

A.3 The tenth-moment construction

The transitive permutation construction in Section 4.8 determines the full tenth-moment expansion. Let $D_i = w_i w_i^*$ with $\sum_i D_i^2 = I$. For $\pi, \rho \in S_5$, let $G_{\pi, \rho}$ have the directed edge multiset

$$E_{\pi, \rho} := \{(a, \pi(a)), (a, \rho(a)) : a \in [5]\}.$$

For an edge occurrence $e = (a, b)$, define the marked expression

$$\mathcal{W}(G_{\pi, \rho}, e) := \sum_{i_1, \dots, i_5=1}^{N_D} \prod_{(c, d) \in E_{\pi, \rho} \setminus \{e\}} (w_{i_c}^* w_{i_d}) w_{i_b} w_{i_a}^*. \tag{248}$$

The multiset difference removes only the chosen occurrence. Then

$$\mathcal{M}_{10} = \frac{1}{240} \sum_{\substack{\pi, \rho \in S_5 \\ \langle \pi, \rho \rangle \text{ transitive}}} \text{sgn}(\pi) \text{sgn}(\rho) \sum_{e \in E_{\pi, \rho}} \mathcal{W}(G_{\pi, \rho}, e). \tag{249}$$

This finite formula follows from the logarithmic determinant expansion and differentiation proved in Section 4.8. To collect its word coefficients, remove self-loop vertices using $\sum_i \|w_i\|^2 w_i w_i^* = I$, joining the remaining edges in their original directions and preserving the marked endpoints. Relabel dummy indices and add equal directed expressions. The 11064 transitive pairs give 110640 marked terms and 95 nonzero collected classes. All Gram expansions in the tenth-moment proof use this same construction.

A.4 Positive Gram expressions

This subsection specifies the maps, ordering, and normalization for all blocks. Definitions A.5 and A.6 specify their coefficient matrices by finite rational constructions.

We first fix the group averaging convention. Let a finite group G act by relabeling the free indices of maps Z_c . Write the canonical maps as $Z_1, \dots, Z_s$ and let each $F(g)$ have r rows and t columns. Suppose $F(g)^*F(h) = K(g^{-1}h)$. Define averaged maps

$$R_{a,b,c} = \frac{1}{|G|} \sum_{g \in G} F(g)_{ab} Z_{gc}.$$

A positive coefficient matrix Q on the pairs (c, b) gives

$$\sum_{a=1}^r \sum_{c,d=1}^s \sum_{b,e=1}^t Q_{(c,b),(d,e)} R_{a,b,c}^* R_{a,e,d} \succeq 0.$$

After summing the free indices and relabeling by g^{-1} , its directed coefficient expansion is

$$\frac{1}{|G|} \sum_{p \in G} \sum_{c,d=1}^s \sum_{b,e=1}^t Q_{(c,b),(d,e)} K(p)_{be} Z_c^* Z_{pd}. \quad (250)$$

For invariant positive diagonal metrics H , the same convention uses row weights H_{aa} and $K(p) = HD(p)$ for an H -orthogonal representation D .

For the trivial and sign blocks, $K(p)$ is respectively 1 and $\text{sgn}(p)$. For the standard block, columns are $b = 1, \dots, k-1$ and

$$K(p)_{be} = \mathbf{1}_{b=p(e)} - \mathbf{1}_{b=p(0)} - \mathbf{1}_{0=p(e)} + \mathbf{1}_{0=p(0)}.$$

For the sign-twisted standard block multiply this by $\text{sgn}(p)$. The pair block for S_4 uses the same difference-column construction on its permutation action on $01|23, 02|13, 03|12$, in that order. The coordinate 0 is again the reference difference column.

The tensor blocks use the following templates and ordering. Use the map L from Lemma 4.32. For $k = 3, 4$, a canonical template is a weak composition (n_0, n_1, n_2, n_3) of k . Order the compositions by nested increasing $n_0, n_1, n_2; n_3$ is the remainder. Assign the successive labels $0, \dots, k-1$ to four ordered lists $\tau_0, \tau_1, \tau_2, \tau_3$ of these lengths. Define

$$Z_\tau = \left(\prod_{a \in \tau_1} D_{i_a} \right) \otimes \left(\prod_{a \in \tau_2} D_{i_a} \right) \otimes \left(\prod_{a \in \tau_3} \overline{D_{i_a}} \right) L \left(\prod_{a \in \tau_0} D_{i_a} \right).$$

The external indices are all independently summed in a Gram product. Empty products are identities, and product orders are the listed orders. The third output factor is conjugate; its directed Gram edges are reversed. Apply (250) to the external-label permutation action. States are ordered first by canonical template, then by difference column.

For **tensor4**, use the trivial, sign, standard, sign-twisted standard, and pair representations, in that order. Their coefficient dimensions are 35, 35, 105, 105, 70. For **tensor3**, use the trivial, sign, and standard representations, with dimensions 20, 20, 40.

The norm blocks are indexed by half words. A half word has length six. For **norm4**, four external labels occur once and one internal label occurs twice. For **norm2**, two external labels occur once and two internal labels each occur twice. The map associated with a half word is the product of its six matrices, summed over internal indices. External indices are fixed until the Gram sum.

If there are s external labels and t internal labels, use the integer alphabets $0, \dots, s-1$ and $s, \dots, s+t-1$, respectively. Canonicalize a half word by first-occurrence relabeling separately in these alphabets, then order the resulting integer tuples lexicographically. This gives 15 templates

for **norm4** and 45 for **norm2**. Apply the group convention to the external-label permutations. For **norm4**, use the trivial, sign, standard, sign-twisted standard, and pair representations, with dimensions 15, 15, 45, 45, 30. For **norm2**, use the trivial and sign representations, both of dimension 45. Internal indices in the two halves of a Gram product are independent.

For **norm6**, all six external labels occur once; the canonical half word is 012345. Use the eleven rational representations of S_6 indexed by the partitions, in descending lexicographic order,

$$(6), (5, 1), (4, 2), (4, 1, 1), (3, 3), (3, 2, 1), (3, 1, 1, 1), (2, 2, 2), (2, 2, 1, 1), (2, 1, 1, 1, 1), (1^6).$$

Their dimensions are

$$1, 5, 9, 10, 5, 16, 10, 5, 9, 5, 1.$$

To specify the rational convention completely, generate standard tableaux recursively by removing an outer corner occupied by the largest label, in increasing row order. The content of a cell is its column index minus its row index. For an adjacent swap s_i , let d be $\text{content}(i+1) - \text{content}(i)$ in tableau T . The matrix has diagonal entry $1/d$ and, when the swapped tableau T' is standard, entry $1 + 1/d$ in row T' , column T ; all other entries in that column are zero. Exact rational multiplication verifies the involution, braid, and distant-commutation relations exactly. Generate all 720 group matrices by breadth-first traversal, starting at the identity and multiplying adjacent generators on the left in increasing order.

Let T_0 be the first tableau in the recursive order just specified. The positive diagonal invariant metric starts with $H_{T_0 T_0} = 1$; along a standard swap its ratio is

$$H_{T' T'} / H_{T T} = (d - 1) / (d + 1).$$

All ratios, positivity, and $HD(s_i) = D(s_i)^* H$ are checked exactly. Formula (250) uses $K(p) = HD(p)$. The corresponding coefficient matrices are named **norm6_0** through **norm6_10**. No square roots or floating-point orthogonalization enter this convention.

The projected blocks use the complementary projection associated with the variance normalization. Let $Vx = (D_i x)_i$ as before. A state word gives a map into this direct sum, with component index i denoted by label 0. Let m_{st} be the number of states in the chosen block. Extra external indices are fixed and summed after forming

$$\sum_{a,b=1}^{m_{\text{st}}} Q_{ab} S_a^* (I - VV^*) S_b \succeq 0.$$

The three blocks are as follows.

- **project_i_k_1** has 36 states: all distinct permutations of $(0, 1, 2, 3, 3)$ with no adjacent pair $(3, 3)$, ordered lexicographically. Labels 0, 1, 2 are shared external indices; label 3 is internally summed.
- **project_i** has 15 states: the distinct permutations of $(0, 1, 1, 2, 2)$, replaced by the lexicographically smaller of the word and its image under exchanging the internal labels 1, 2, then deduplicated and ordered lexicographically. Label 0 is external.
- **project_i_k** has the five ordered state words $(1, 2, 0, 2)$, $(2, 0, 1, 2)$, $(2, 0, 2, 1)$, $(2, 1, 0, 2)$, $(2, 1, 2, 0)$. Labels 0, 1 are external and 2 internal.

In the identity part of the projection, all external labels are shared across the two halves and internal labels are independent. In the VV^* part, prepend label 0 to each state word; the left and right sums over that label are now independent, as are their internal sums. Only the other external labels remain shared. This fixes the directed expansion of each projected block.

The remaining blocks are polynomial Gram expressions and scalar positive generators. The 3×3 block WEI, with states $[W, \mathcal{E}, I]$, means

$$[W, \mathcal{E}, I]^* Q[W, \mathcal{E}, I] + Q_{11} \mathcal{E}(I - \mathcal{E}) \succeq 0,$$

where the $\mathcal{E}$ state is indexed by 1 in zero-based numbering. Its coefficient expansion is

$$Q_{00}W^2 + Q_{01}(W\mathcal{E} + \mathcal{E}W) + 2Q_{02}W + (Q_{11} + 2Q_{12})\mathcal{E} + Q_{22}I.$$

The 5×5 block `polynomial15` is the ordinary positive Gram expression on the Hermitian states $[I, W, \Psi(W), A_4, Y]$.

The thirteen nonnegative scalar coefficients multiply, in order,

$$\begin{aligned} &R_0, W, \mathcal{E}, I - W, I - \mathcal{E}, \\ &\mathcal{E} - \mathcal{E}^2, \Psi(R_0), \Psi^2(R_0), \Psi(R_*), \\ &\Psi(W^2) - \Psi(W)^2, \Psi(W) - \Psi(W)^2, A_4 - A_4^2, Y - Y^2. \end{aligned}$$

They are positive by Lemma 4.32, positivity of Ψ , and its Schwarz inequality. Here $\Psi(R_*)$ uses the coefficient 1/3 in Eq. (149).

The 31 Gram blocks have total coefficient dimension 810. Definitions A.5 and A.6 construct their coefficient matrices by polynomial factors. Lemma A.15 gives the required positivity and residual bounds.

A.5 Polynomial Gram constructions

We construct the three Gram witnesses by a common matrix-polynomial recurrence. Algorithm 1 generates their rational seed vectors from the directed moment coefficient maps.

The directed contraction rules fix both the coefficient maps and the order of the seeds.

Definition A.1 (Directed coefficient maps). *For a family of real symmetric matrices $Q = (Q_b)_b$, let $\mathcal{A}(Q)$ be the directed coefficient vector of its Gram expression. For an upper-triangular coordinate (b, i, j) , let $a_{w,b,i,j}$ be its coefficient in directed class w ; for $i < j$, this includes both ordered Gram products. Set $\omega_{ij} := 1$ for $i = j$ and $\omega_{ij} := 2$ for $i < j$.*

After eliminating self-loops, let g be the adjacency matrix of a directed graph on n vertices with marked edge (a, b) . Its canonical code is

$$n2^{60} + \mathbf{1}_{a=b}2^{59} + \min_{\pi} \sum_{u,v=0}^{n-1} g_{\pi(u),\pi(v)} 3^{n^2-1-un-v},$$

where the permutations π start with a when $a = b$, and with (a, b) otherwise. Order classes by increasing code. For the tenth-moment map, use all classes occurring in the two Gram expansions. For the twelfth-moment maps, use all 588 pair-word classes through order twelve and retain the columns of the varying blocks specified below. Delete zero rows, and retain the first copy of every distinct remaining row. Denote this reduced coefficient map by B .

Use the direct-sum Frobenius inner product on block matrices and the Euclidean inner product on coefficient vectors. The adjoint is then

$$(B^*\theta)_{b,ij} := \frac{1}{\omega_{ij}} \sum_k a_{k,b,i,j} \theta_k.$$

The seed coordinates start at zero and follow the retained row order.

The polynomial recurrence supplies positive factors without listing their individual matrix entries.

Definition A.2 (Polynomial Gram factors). Define $s_0(t) := t$ and

$$s_{r+1}(t) := \frac{3s_r(t) - s_r(t)^3}{2}. \quad (251)$$

For a seed vector θ , set $H := B^*\theta$. For each block, choose the smallest integer $e_b \geq 0$ satisfying

$$2^{e_b} \geq \max_i \sum_j |(H_b)_{ij}|,$$

and set $R_b := 2^{e_b}$. Given an integer $m \geq 0$ and $\varepsilon \geq 0$, define

$$F_b := \frac{1}{2} H_b [I + s_m(H_b/R_b)], \quad \widehat{Q}_b := \varepsilon I + F_b^\top F_b. \quad (252)$$

Every seed below has the form $\theta_k := n_k/10^{12}$.

The recurrence maps the spectral interval into itself. This gives the norm bounds needed for exact evaluation.

Lemma A.3 (Polynomial factor bounds). Suppose H_b is real symmetric with $\|H_b\| \leq R_b$. Then $\|s_r(H_b/R_b)\| \leq 1$ for every $r \geq 0$, $\|F_b\| \leq R_b$, and $\widehat{Q}_b \succeq \varepsilon I$.

Proof. The polynomial $(3t - t^3)/2$ maps $[-1, 1]$ into itself: its derivative is $3(1 - t^2)/2$, and its end-point values are -1 and 1 . Induction and the spectral theorem give the first bound. Equation (252) gives the factor norm bound and positivity. $\square$

For the tenth moment, a rational correction enforces the coefficient identity exactly.

Definition A.4 (Tenth-moment coefficient matrices). Use the 120 tensor states and five projected states of Eq. (138), in their stated order. For these two blocks, B has 69 rows. Use the seed vector returned by Algorithm 1 in case 10, $m := 64$, and $\varepsilon := 1/500000$ in Definition A.2. Let t_{10} be the retained coefficient vector of

$$-\frac{39}{25}I - \mathcal{M}_{10} + \frac{141}{25}\mathcal{M}_8 - \frac{59}{5}\mathcal{M}_6 + \frac{59}{5}\mathcal{M}_4 - R_3 - \frac{109}{25}R_0.$$

Set $M := BB^*$ and define the two blocks (Q, P_5) by

$$(Q, P_5) := \widehat{Q} + B^* M^{-1} [t_{10} - B(\widehat{Q})]. \quad (253)$$

Here M is a fixed rational matrix of order 69; its invertibility is verified in Lemma A.15. Equivalently, $M^{-1} = \text{adj}(M)/\det M$.

The next witness uses the 31 positive Gram maps of Section A.4.

Definition A.5 (Twelfth-moment coefficient matrices). *Index the 31 positive Gram blocks defined in Appendix A.4 in the following order.*

Indices b	Blocks (in order)
1, 2, 3	tensor4_trivial, tensor4_sign, tensor4_standard
4, 5	tensor4_std_sign, tensor4_pair
6, 7, 8	tensor3_trivial, tensor3_sign, tensor3_standard
9, 10, 11	project_i_k_l, project_i, project_i_k
12, 13	WEI, polynomial5
14, 15, 16	norm4_trivial, norm4_sign, norm4_standard
17, 18	norm4_std_sign, norm4_pair
19, 20	norm2_trivial, norm2_sign
21, ..., 31	norm6_0, ..., norm6_10

Let n_b be the dimensions in Section A.4. Vary the fifteen blocks with indices

$$1, 2, 3, 4, 5, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20.$$

For these columns, B has 395 rows. Use the seed vector returned by Algorithm 1 in case 12, $m := 96$, and $\varepsilon := 1/5000000000$ in Definition A.2. Use $\tilde{Q}_b$ in these blocks and $10^{-8}I_{n_b}$ in the remaining sixteen blocks. Set all thirteen scalar weights to zero. Denote the resulting positive expression by $\mathcal{P}_{\text{cert}}$.

The final witness uses polynomial factors for thirteen active blocks.

Definition A.6 (Factored twelfth-moment coefficients). *Use the thirteen blocks in the order of Lemma 4.44, with the coefficient dimensions n_b specified there. For these columns, B has 395 rows. Use the seed vector returned by Algorithm 1 in case F, $m := 96$, and $\varepsilon := 0$ in Definition A.2. Use $Q_b := F_b^\top F_b$ in the corresponding Gram maps. Every factor F_b is square of order n_b . All other block coefficients and scalar weights are zero. Denote the resulting positive expression by $\mathcal{P}_F$.*

The polynomial values can be checked by finite integer recurrences. We specify the rounding and coefficient-sum calculation explicitly.

Definition A.7 (Rational evaluation). *Fix one of Definitions A.4–A.6. Let n_b be the order of its varying block b , let p be a positive integer, and set $N := 2^p$. Every floor in this definition acts entrywise, including on negative entries. For each varying block define*

$$S_{b,0} := \lfloor NH_b/R_b \rfloor, \quad (254)$$

$$U_{b,r} := \lfloor S_{b,r}^2/N \rfloor, \quad (255)$$

$$V_{b,r} := \lfloor U_{b,r}S_{b,r}/N \rfloor, \quad (256)$$

$$S_{b,r+1} := \lfloor (3S_{b,r} - V_{b,r})/2 \rfloor \quad (0 \leq r < m), \quad (257)$$

$$K_b := \lfloor H_b(NI + S_{b,m})/2 \rfloor, \quad (258)$$

$$\tilde{Q}_b := \varepsilon I + N^{-2}K_b^\top K_b. \quad (259)$$

Thus $S_{b,r}, U_{b,r}, V_{b,r}, K_b$ are integer matrices, and $\tilde{Q}_b$ is rational. Define

$$e_b := \frac{(2 \cdot 4^m - 1)n_b}{N}, \quad (260)$$

$$f_b := \frac{R_b e_b}{2} + \frac{n_b}{N}, \quad (261)$$

$$\eta_b := 2R_b f_b + f_b^2. \quad (262)$$

Let $\mathcal{C}$ be a chosen row restriction of the coefficient map on the varying blocks, and let $c_{w,b,i,j}$ be its upper-triangular coefficients. For a target vector t , define the rational number

$$\mathfrak{U}(\mathcal{C}, t) := \sum_w |t_w - \mathcal{C}(\tilde{Q})_w| + \sum_{w,b,i \leq j} |c_{w,b,i,j}| \eta_b. \quad (263)$$

For the tenth moment, take $(\mathcal{C}, t) := (B, t_{10})$. For the first twelfth-moment witness, use all 588 rows of the varying blocks and the directed coefficient vector of $\mathcal{T}_0$ after subtracting the sixteen fixed $10^{-8}I$ block expressions. For the final witness, use all 588 rows and the directed coefficient vector of

$$-\frac{16510788487}{5000000000}I - \mathcal{M}_{12} + g_n \mathcal{M}_{10} - h_n \mathcal{M}_8 + j_n \mathcal{M}_6 - \ell_n \mathcal{M}_4.$$

The next estimate relates the integer recurrences to the exact polynomial matrices used in the proof.

Lemma A.8 (Rational polynomial enclosures). *In Definition A.7, suppose $\|H_b\| \leq R_b$ and $e_b < 1/2$ for every varying block. Then*

$$\|\hat{Q}_b - \tilde{Q}_b\| \leq \eta_b, \quad \|t - \mathcal{C}(\hat{Q})\|_1 \leq \mathfrak{U}(\mathcal{C}, t).$$

Proof. For a block of order n , entrywise downward rounding to multiples of $1/N$ has Frobenius norm, and hence operator norm, at most n/N . Let X be the approximate polynomial value and S its exact value. Lemma A.3 gives $\|S\| \leq 1$. If $\|X - S\| \leq e$, then $\|X^3 - S^3\| \leq 3e + 3e^2 + e^3$. The cubic update therefore has perturbation at most $3e + 3e^2/2 + e^3/2 \leq 4e$ for $e < 1/2$. The two rounded products and final rounding contribute at most $3n/N$. Starting with error n/N , induction gives $(2 \cdot 4^r - 1)n/N$ after r steps. Consequently, the error in K_b/N is at most f_b . Expanding the difference of the two Gram matrices and using $\|F_b\| \leq R_b$ gives the bound η_b .

Every entry error is at most the operator norm error. Apply the coefficient map to $\hat{Q} - \tilde{Q}$, multiply each entry bound by $|c_{w,b,i,j}|$, and sum over the directed classes. The triangle inequality then gives Eq. (263) as an upper bound for the exact residual coefficient sum. $\square$

A.6 Generation of the Gram coefficients

The following finite recurrence determines the seed vectors from the moment coefficient maps. Its rounding rules are part of the definition. The resulting rational polynomial factors are then checked by Definition A.7.

We first specify the scalar arithmetic and the order of every reduction.

Definition A.9 (Rounded arithmetic). *Let $\mathbb{F}$ consist of zero, the numbers $\pm n2^{-1074}$ with $1 \leq n < 2^{52}$, and the numbers $\pm n2^e$ with $2^{52} \leq n < 2^{53}$ and $-1074 \leq e \leq 971$, where n, e are integers. Let ρ round to the nearest element of $\mathbb{F}$, breaking a tie by the even significand. A magnitude at least $2^{1024} - 2^{970}$, a zero divisor, or a square root of a negative number returns failure. In Definitions A.11–A.13 and Algorithm 1, round every scalar addition, subtraction, multiplication, division, and square root by ρ . Round rational constants before their use. Explicitly designated exact computations are excepted. Signs, absolute values, comparisons, and transposes are exact.*

A dot product starts at zero and accumulates separately rounded products in increasing index order. Every matrix product uses this dot product, without combining a multiplication and an addition into one rounding. Nested products are evaluated in the displayed parentheses. A sum over matrix entries uses row order, with the column index varying fastest. Norms use these ordered sums of squares followed by a rounded square root. Denote this matrix product by $\star$.

The rounding operations themselves admit finite arithmetic evaluation.

Lemma A.10. *Each operation of Definition A.9 on finite inputs is determined by integer comparisons, with an explicit failure outcome when an operation is undefined.*

Proof. For integers $a \geq 0$ and $b > 0$, define

$$q := \lfloor a/b \rfloor, \quad r := a - bq, \quad \nu(a, b) := q + \mathbf{1}_{\{2r > b\} \cup \{2r = b, q \text{ odd}\}}.$$

Thus $\nu(a, b)$ is nearest-integer rounding of a/b , with ties to even. Every division in this formula is an integer quotient.

For a nonzero rational x , first perform the overflow comparison in Definition A.9. Choose the integer h by $2^h \leq |x| < 2^{h+1}$ and set $e := \max\{-1074, h - 52\}$. Express $|x|2^{-e} = a/b$ with positive integers a, b . Then

$$\rho(x) = \text{sgn}(x)2^e\nu(a, b), \quad \rho(0) = 0.$$

The grid spacing on $[2^h, 2^{h+1})$ is 2^e ; below the smallest normal number it is 2^{-1074} . At an endpoint, the two formulas select the same even endpoint. This proves the formula, including subnormal values. The integer h is found by comparisons with powers of two, using repeated doubling or halving.

For a positive rational x , rounded square root is equally explicit. Choose h by $2^{2h} \leq x < 2^{2h+2}$ and set $e := \max\{-1074, h - 52\}$. Express $x2^{-2e} = a/b$ and let q be the largest nonnegative integer satisfying $bq^2 \leq a$. Set

$$d := 4a - b(2q + 1)^2, \quad \rho(\sqrt{x}) = 2^e[q + \mathbf{1}_{\{d > 0\} \cup \{d = 0, q \text{ odd}\}}].$$

The sign of d compares $\sqrt{a/b}$ with the midpoint $q + 1/2$. The value q is obtained by integer bisection between 0 and $a + b + 1$. The cases $x = 0$ and $x < 0$ give zero and failure, respectively.

Addition, subtraction, multiplication, and division of dyadic rationals are rational operations, so the first formula applies to each of them. The displayed square-root rule handles the remaining scalar operation. Cross-multiplication by positive denominators implements every branch comparison. These rules give a finite integer evaluation of every operation and of each failure test. $\square$

The three inputs differ only in their block families and moment targets.

Definition A.11 (Generation inputs). *Use the varying blocks of Definitions A.4, A.5, and A.6, respectively. Let A be the full coefficient array and B its reduced array from Definition A.1. Order the upper-triangular coordinates $\alpha = (b, i, j)$ by b , then i , then j . Let t be the full target: the expression in Definition A.4 for the tenth moment; $\mathcal{T}_0$ of Eq. (152), after subtracting the sixteen fixed blocks, for the first twelfth moment; and the final target displayed in Definition A.7. All scalar weights are zero in the two twelfth-moment cases.*

In the case order (10, 12, F), set

$$(\eta_{10}, \eta_{12}, \eta_F) := (0, 487/25000, 9/25000), \quad (\tau_{10}, \tau_{12}, \tau_F) := (10^{-9}, 10^{-7}, 10^{-7}).$$

Use the respective values of ε in Definitions A.4–A.6. Form the following quantities in exact rational arithmetic before rounding their entries once:

$$u := \rho(t + \eta e_I - \varepsilon A(I)), \quad M := \rho(BB^*), \quad B_0 := \rho(B).$$

Here e_I is the identity coefficient vector, and I denotes the identities on the varying blocks. Let u_B retain the rows used in B . Construct the lower-triangular array L in increasing row and column order by

$$L_{ij} := \frac{M_{ij} - \sum_{k < j} L_{ik} L_{jk}}{L_{jj}} \quad (i > j), \quad L_{ii} := \sqrt{M_{ii} - \sum_{k < i} L_{ik}^2}.$$

Set $L_{ij} := 0$ for $i < j$. A nonpositive diagonal radicand returns failure. Construct J column by column by $J_{ij} := 0$ for $i < j$ and

$$J_{ij} := \frac{\mathbf{1}_{i=j} - \sum_{k < i} L_{ik} J_{kj}}{L_{ii}} \quad (i \geq j).$$

Set $C := J \star B_0$ and $v := J \star u_B$. For each retained row a , define the symmetric block arrays $(E_a)_{b,ij} := C_{a,(b,i,j)}/\omega_{ij}$. For every full row w , record the retained row $\iota(w)$ equal to it; a zero row has no retained index.

A fixed sequence of rotations supplies the arrays used in each update.

Definition A.12 (Cyclic rotations). Given a symmetric array H of order n , set $W := H$ and $U := I_n$. Perform at most 32 sweeps. In each sweep visit (p, q) with $0 \leq p < q < n$ in increasing p , then increasing q . Put $a := W_{pp}$, $b := W_{pq}$, and $d := W_{qq}$. Skip the pair if $|b| \leq 2^{-54}[(1 + |a|) + |d|]$. Otherwise set

$$z := \frac{d - a}{2b}, \quad h := \frac{\text{sgn}_+(z)}{|z| + \sqrt{1 + z^2}}, \quad c := \frac{1}{\sqrt{1 + h^2}}, \quad s := hc,$$

where $\text{sgn}_+(z) := 1$ for $z \geq 0$ and -1 otherwise. Set $W_{pp} := a - hb$, $W_{qq} := d + hb$, and $W_{pq} := W_{qp} := 0$. For $k \notin \{p, q\}$, use the old pair $(x, y) := (W_{kp}, W_{kq})$ to set

$$W_{kp} = W_{pk} := cx - sy, \quad W_{kq} = W_{qk} := sx + cy.$$

Apply the same two-coordinate update to the old pair (U_{kp}, U_{kq}) for every k . Stop after a sweep that changes no pair. Return $a_i := W_{ii}$ and U , without sorting the diagonal entries.

The next definition fixes both the evaluation and the regularized update matrix; each formula uses the rounded arithmetic already specified.

Definition A.13 (Generation evaluation). Given y , form the symmetric blocks

$$H_{b,ij} := \frac{\sum_a C_{a,(b,i,j)} y_a}{\omega_{ij}}.$$

Apply Definition A.12 separately to each block, in block order. With its returned arrays, put $p_i := \max\{a_i, 0\}$, $V_{ij} := U_{ij} p_j^2$, and $Q := V \star U^\top$. Retain $q_{(b,i,j)} := Q_{ij}$ for $i \leq j$. Start f at $-\sum_a v_a y_a$ and, for each block in order, add $[\sum_i (p_i^2) p_i]/3$. Set $g := C \star q - v$. Compute $w := B_0 \star q$ and

$$r := \sum_{\text{full rows } j} |w_{\iota(j)} - u_j|,$$

using zero in place of $w_{\iota(j)}$ on a zero row.

For each block, form the symmetric array

$$G_{ij} := \begin{cases} a_i + a_j, & a_i, a_j \geq 0, \\ a_i^2/(a_i - a_j), & a_i > 0 > a_j, \\ a_j^2/(a_j - a_i), & a_j > 0 > a_i, \\ 0, & \text{otherwise.} \end{cases}$$

For each retained row a , let row a of T_b be the row-ordered entries of $[(U^\top \star (E_a)_b) \star U] \circ (\sqrt{G_{ij}})_{ij}$, where $\circ$ is entrywise multiplication. Start K at zero and add $T_b \star T_b^\top$ in block order. Return (f, g, K, r) . Evaluations used only for line selection need only return f and g .

To solve a linear system of order N in Algorithm 1, use Gaussian elimination. At column n , choose a row of maximum absolute pivot among rows $k, \dots, N-1$, breaking ties by the smallest row index, and swap it into row k . For each row $i > k$, compute the multiplier a_{ik}/a_{kk} , set $a_{ik} := 0$, and subtract the multiplier times row k from the later columns and the right side. Use increasing column order. Back substitution uses increasing column order in each dot product. A zero pivot returns failure.

Algorithm 1 starts from the zero vector and uses only the moment and Gram maps defined in this paper. The next lemma verifies its three bounded runs. All numerical checks in this subsection are finite computations; the integer rules in Lemma A.10 specify their arithmetic.

Lemma A.14 (Finite generation). *Algorithm 1 returns seed vectors of lengths 69, 395, 395, respectively, after 18, 46, 84 updates. Every divisor used in these runs is nonzero, and every required square root is defined.*

Proof. Step 1: Determine the arithmetic inputs. Expand the directed coefficient maps by Definition A.1. Retain the first occurrence of each nonzero row in the prescribed order; this gives 69, 395, 395 rows. The target and every entry of BB^* are rational expressions in these maps. Evaluate them exactly, and apply the integer rounding formulas of Lemma A.10. This determines the initialization of Definition A.11.

Step 2: Check each bounded run. Starting at $y = 0$, evaluate Algorithm 1 in its stated order. Before each division check its denominator, and before each square root check its radicand. Apply the overflow test before rounding. Each comparison is an integer comparison by Lemma A.10. Record the first successful outer index and the largest accepted inner index. The resulting index pairs, in the order (10, 12, F), are

$$(18, 32), \quad (46, 33), \quad (84, 33).$$

Every initial diagonal radicand exceeds $1/32$, and the absolute value of every elimination pivot exceeds 2^{-34} . All remaining scalar operations pass their domain and overflow tests. The residuals at the first successful outer indices satisfy

$$r_{10} < \frac{3}{10^{12}}, \quad r_{12} < \frac{19}{500000000}, \quad r_F < \frac{1}{12500000}.$$

These are strict rational comparisons with the dyadic outputs of the recurrence. In particular, each is below its prescribed stopping tolerance, and every accepted inner index is below 60.

Step 3: Form the rational seeds. For a coordinate $z_a = \text{sgn}(z_a)u/v$ with integers $u \geq 0$ and $v > 0$, the last line of the construction is

$$n_a := \text{sgn}(z_a)\nu(10^{12}u, v), \quad \theta_a := n_a/10^{12}.$$

Here ν is the integer rule in Lemma A.10; for $z_a = 0$, set $n_a := 0$. Thus the generated seeds are exact rational numbers determined by the stated inputs and bounded recurrences. The successful stopping indices in Step 2 establish all three asserted returns. $\square$

Algorithm 1 uses exact nearest-integer rounding nint , with ties to even, in its return step.

Algorithm 1 Generation of the Gram seeds

```

1: procedure GRAMSEEDS(one of the cases 10, 12, F)
2:   Form  $B_0, C, J, u, v, E_a$  and  $\tau$  by Definition A.11
3:    $y \leftarrow 0$ 
4:   for  $k = 0, \dots, 128$  do
5:      $(f, g, K, r) \leftarrow \text{EVALUATE}(y)$  ▷ Definition A.13
6:     if  $r < \tau$  then
7:        $z \leftarrow J^\top \star y$ 
8:        $\theta_a \leftarrow 10^{-12} \text{nint}(10^{12} z_a)$  for every  $a$  ▷ exact arithmetic; ties to even
9:       return  $\theta$ 
10:    end if
11:    if  $k = 128$  then
12:      return failure
13:    end if
14:     $\lambda \leftarrow \max\{10^{-10}, 10^{-12} \|K\|_F\}$ 
15:    Solve  $(K + \lambda I)d = -g$  by Definition A.13
16:     $\sigma \leftarrow \sum_a g_a d_a$  and  $\alpha \leftarrow 1$ 
17:    for  $j = 0, \dots, 59$  do
18:       $y' \leftarrow y + \alpha d$  and  $(f', g') \leftarrow \text{EVALUATE}(y')$ 
19:      if  $f' \leq f + (10^{-4} \alpha) \sigma$  or  $\|g'\|_2 \leq (4/5) \|g\|_2$  then
20:         $y \leftarrow y'$ ; exit the inner loop
21:      end if
22:       $\alpha \leftarrow \alpha/2$ 
23:    end for
24:    if no inner-loop test succeeded then
25:      return failure
26:    end if
27:  end for
28: end procedure

```

The next proof evaluates the generated seeds by the integer matrix recurrences of Definition A.7. It specifies the residual comparisons and derives positivity and the coefficient identities from them. Algorithm 2 collects the verification steps using only the case index as input.

Lemma A.15 (Gram verification). *The matrices of Definition A.4 satisfy Eq. (138) coefficientwise. Their smallest eigenvalues exceed $99/500000000$, and their $LDL^\top$ pivots exceed 10^{-6} .*

Every Gram pivot in Definition A.5 exceeds 10^{-10} , and its scalar weights are zero. With $\mathcal{T}_0$ as in Eq. (152), the residual $\mathcal{T}_0 - \mathcal{P}_{\text{cert}}$ is Hermitian, is supported on the 588 directed pair-word classes, and has coefficient sum less than $39/2000 < 49/2500$.

For Definition A.6, put $\kappa_0 := -\frac{16510788487}{5000000000}$. The residual

$$\kappa_0 I - \mathcal{M}_{12} + g_n \mathcal{M}_{10} - h_n \mathcal{M}_8 + j_n \mathcal{M}_6 - \ell_n \mathcal{M}_4 - \mathcal{P}_F$$

is Hermitian, is supported on the same directed classes, and has coefficient sum less than $3/8000 < 1/2500$.

Proof. Step 1: Form the directed coefficient maps. Expand the moments by the transitive permutation-pair rule, with sign $(-1)^{k+1}/(2k!)$ after marking an edge. At orders two through twelve, the resulting numbers of directed classes are 1, 2, 5, 19, 95, 588. Expand the Gram maps by Eq. (250) and the projection rules of Section A.4, preserving every edge direction. Order the rows and form B by Definition A.1. The two tenth-moment blocks give 87 rows and 69 distinct rows; both twelfth-moment maps have 395 distinct nonzero rows among the 588 classes. The tenth-moment coefficients are integers, and multiplying the varying twelfth-moment coefficient maps by 24 gives integers.

Step 2: Evaluate rational residual bounds. Form $H = B^*\theta$ from the seeds of Algorithm 1. Its entries can be expressed with denominator $2 \cdot 10^{12}$ for the tenth moment and $48 \cdot 10^{12}$ for each twelfth-moment witness. The definition of R_b gives $\max_i \sum_j |(H_b)_{ij}| \leq R_b$ for each block. Evaluate Eqs. (254)–(259) with $(m, p) = (64, 256)$ for the tenth moment and $(m, p) = (96, 384)$ for each twelfth-moment witness.

The following integer calculation evaluates the residual sum explicitly. Let $d := 1$ in the tenth-moment case and $d := 24$ otherwise, so that $a_{w,b,i,j} := d c_{w,b,i,j}$ is integral. Express $\varepsilon = \alpha/\beta$ with coprime nonnegative integers α, β and $\beta > 0$; use $(\alpha, \beta) = (0, 1)$ when $\varepsilon = 0$. Define

$$Z_b := \beta K_b^\top K_b + \alpha N^2 I, \quad L := \text{lcm}_w \text{den}(t_w), \quad T := d\beta N^2 L.$$

Here $\text{den}(t_w)$ is the positive denominator in lowest terms. Set

$$v_w := d\beta N^2 (Lt_w) - L \sum_{b,i \leq j} a_{w,b,i,j} (Z_b)_{ij}, \quad R := \sum_w |v_w|.$$

Every v_w is an integer. Since $\tilde{Q}_b = Z_b/(\beta N^2)$, the exact residual coefficient sum for $\tilde{Q}$ equals R/T . Let $\Delta := \sum_{w,b,i \leq j} |c_{w,b,i,j}| \eta_b$, with η_b as in Definition A.7. Reduce $R/T + \Delta$ to a fraction a/b with integers $a \geq 0$, $b > 0$, and compute the integer

$$u := \lfloor (10^{12}a + b - 1)/b \rfloor.$$

This is precisely $\lceil 10^{12} \mathfrak{U}(\mathcal{C}, t) \rceil$. The integer recurrences give

$$(u_{10}, u_{12}, u_F) = (758, 19480038250, 360092344).$$

The accompanying rational error bounds satisfy $e_b < 1/2$ for every block and $\Delta < 10^{-20}$ in each case. Therefore

$$\begin{aligned} \mathfrak{U}(B, t_{10}) &\leq \frac{758}{10^{12}} < \frac{1}{10^8}, \\ \mathfrak{U}(\mathcal{C}, t) &\leq \frac{19480038250}{10^{12}} < \frac{39}{2000} && \text{in case 12,} \\ \mathfrak{U}(\mathcal{C}, t) &\leq \frac{360092344}{10^{12}} < \frac{3}{8000} && \text{in case F.} \end{aligned}$$

Each last comparison follows by integer cross-multiplication. Lemma A.8 transfers these bounds from the integer recurrences to the exact polynomial matrices.

Step 3: Enforce the tenth-moment identity. Form $M := BB^*$. To compute its inverse, start with $[M \mid I]$. At column k , use the first nonzero pivot in rows $k, \dots, 68$, swap it into row

k , divide that row by the pivot, and eliminate column k from every other row. All 69 pivots are nonzero. The resulting right block J satisfies $MJ = I$ and $\max_i \sum_j |J_{ij}| < 2$, by rational multiplication and comparison. Hence $J = M^{-1}$ and $\|M^{-1}\|_2 < 2$, since M^{-1} is symmetric.

Put $r := t_{10} - B(\hat{Q})$ and $E := B^*M^{-1}r$. Step 2 and Lemma A.8 give $\|r\|_1 < 10^{-8}$. Since $B(E) = r$, Eq. (253) enforces every retained coefficient. The omitted rows repeat retained rows; direct comparison of the target coefficients gives the same equalities for the target. Thus the full directed identity holds. Moreover, $\|E\|_F^2 = r^T M^{-1} r$, so $\|E\|_F < 2\|r\|_1 < 2 \times 10^{-8}$.

Step 4: Prove positivity. Every corrected tenth-moment block has smallest eigenvalue greater than $1/500000 - 2 \times 10^{-8} = 99/500000000$. For the first twelfth-moment witness, every varying block is at least $I/5000000000$, and every fixed block equals $10^{-8}I$. For a positive matrix bounded below by cI , every successive Schur complement is bounded below by cI : its quadratic form is obtained by minimizing over the eliminated coordinates. This proves the stated pivot bounds. The final witness is positive semidefinite by its factorization.

Step 5: Bound the twelfth-moment residuals. Every real symmetric coefficient block defines a Hermitian Gram expression: taking the adjoint exchanges its two state indices. Equivalently, reversing a directed word and exchanging i, j leaves the sum of the two off-diagonal Gram terms unchanged; the diagonal terms are Hermitian. The moment targets are Hermitian as well. Thus both exact residuals are Hermitian. Every class has a pair-word representative by the contraction rules in Section 4.9. Step 2 and Lemma A.8 bound the exact coefficient sums by $39/2000$ and $3/8000$, respectively. Finally, $39/2000 < 49/2500$ and $3/8000 < 1/2500$, giving the required comparisons with the repair allowances. $\square$

The following algorithm restates the finite checks used in Lemma A.15. All arithmetic after seed generation is integer or rational arithmetic. The polynomial matrices are generated inside the procedure from the displayed definitions and recurrences. Exact elimination uses the first nonzero pivot in the current column, swaps it into the diagonal, divides the pivot row by that pivot, and eliminates the column from all other rows. If no pivot exists, the procedure returns false immediately.

The finite certificate checks in Section 4 and this appendix were performed computationally using the arithmetic procedures specified in the paper. In particular, Algorithms 1 and 2 generate and verify the Gram certificates for Lemmas A.14 and A.15. The displayed definitions and recurrences specify every input, coefficient, seed, rounding rule, and verification condition needed to reproduce these calculations.

A.7 Twelfth-moment schedule polynomials

For stage $j = 1, 2, 3$ in Eq. (171), define $a_i := [x^i]P_j^{(12)}(x)$. The schedule is $k(s) = 10^{-12}P_j^{(12)}(x)$, where

$$\begin{aligned} P_1^{(12)}(x) := & 1023470309130 + 82113606874x - 1418890078x^2 \\ & + 37520378027x^3 - 595493629118x^4 + 3042773040497x^5 \\ & - 7596614041767x^6 + 9985474790892x^7 - 6608620880836x^8 \\ & + 1734607013635x^9. \end{aligned} \tag{264}$$

$$\begin{aligned} P_2^{(12)}(x) := & 1024526034971 + 76718030404x - 54478850531x^2 \\ & + 712573818283x^3 - 4568807288298x^4 + 15267812411736x^5 \\ & - 28214562186215x^6 + 29233359009948x^7 - 15950755921098x^8 \\ & + 3574191158483x^9. \end{aligned} \tag{265}$$

Algorithm 2 Verification of a generated Gram construction

```

1: procedure VERIFYGRAM( $c \in \{10, 12, F\}$ )
2:    $\theta \leftarrow \text{GRAMSEEDS}(c)$  ▷ Algorithm 1
3:   if generation returned failure then
4:     return false
5:   end if
6:   Form  $\mathcal{C}, t, B$  by Definitions A.1 and A.7
7:    $H \leftarrow B^* \theta$ ; compute  $R_b$  by Definition A.2
8:    $(m, p) \leftarrow (64, 256)$  if  $c = 10$ ; otherwise  $(m, p) \leftarrow (96, 384)$ 
9:   Compute  $S_{b,r}, U_{b,r}, V_{b,r}, K_b, Q_b, e_b, f_b, \eta_b$  for every block ▷ Definition A.7
10:   $U \leftarrow \mathfrak{U}(\mathcal{C}, t)$  and  $\Delta \leftarrow \sum_{w,b,i \leq j} |c_{w,b,i,j}| \eta_b$ 
11:   $u \leftarrow \lceil 10^{12} U \rceil$ 
12:   $u_* \leftarrow 758, 19480038250, 360092344$  in cases 10, 12, F, respectively
13:  if  $u \neq u_*$  or some  $e_b \geq 1/2$  or  $\Delta \geq 10^{-20}$  then
14:    return false
15:  end if
16:  if  $c = 10$  then
17:     $M \leftarrow BB^*$ ; form  $J$  by exact elimination on  $[M \mid I]$ 
18:    Check  $MJ = I$ ,  $J = J^T$ , and  $\max_i \sum_j |J_{ij}| < 2$ 
19:    Check that equal rows of the full coefficient map have equal targets
20:    if a pivot is zero or any check fails then
21:      return false
22:    end if
23:  end if
24:  return true
25: end procedure

```

$$\begin{aligned}
P_3^{(12)}(x) := & 1024956978960 + 69493568722x + 110408902264x^2 \\
& - 1287879182125x^3 + 6773229724061x^4 - 18778275562230x^5 \\
& + 29814585994077x^6 - 27377921066164x^7 + 13557209228843x^8 \\
& - 2806315046791x^9.
\end{aligned} \tag{266}$$

The cancellations in this power basis are handled by exact rational arithmetic.

A.8 Resolvent majorants

The following rational construction specifies the twenty constraints and two initial barriers used in Section 4.12. Entry 0 is the seed in Eq. (197). Each subsequent entry specifies Eq. (194). The indices u_1, u_2 specify the initial barriers for the two schedules and are not inserted as additional resolvent constraints. Their arguments are $a = m_0/s_0$.

Use $C = 43919/20000$ for entries $0, \dots, 11, u_1$ and $C = 5453/2500$ for entries $12, \dots, 19, u_2$. All

parameters below are exact rational numbers. The arguments are generated by

$$\begin{aligned}
a_0 = a_1 &:= 56/25, & a_2 &:= 111/50, \\
(a_{3+3h}, a_{4+3h}, a_{5+3h}) &:= (221/100, 11/5, 2197/1000) & (h = 0, 1), \\
a_9 &:= 11/5, & a_{10} = a_{11} &:= 2197/1000, \\
(a_{12+3h}, a_{13+3h}) &:= (2739/1250, 5463/2500) & (h = 0, 1, 2), \\
a_{14} = a_{17} &:= 10911/5000.
\end{aligned}$$

The corresponding bounds are

$$\begin{aligned}
(10^6 K_0, \dots, 10^6 K_{19}) &:= (869556, 868341, 940210, 961030, 1030737, \\
&1044451, 932891, 971563, 970588, 944129, \\
&953410, 949006, 976057, 1011682, 1045461, \\
&971563, 1006966, 1036610, 968592, 1003822).
\end{aligned}$$

For the two initial barriers, set

$$\begin{aligned}
(a_{u_1}, K_{u_1}) &:= (197233/89727, 942796/10^6), \\
(a_{u_2}, K_{u_2}) &:= (99322/45271, 953521/10^6).
\end{aligned}$$

The construction uses $c_4 = c_5 = 0$. The remaining multipliers have denominator 10^{12} , and

$$A = \lambda_0 + \lambda_1 + 2c_2 - 3c_3 + \kappa c_6, \quad B = c_2, \quad \kappa = -18223989/2500000.$$

The seed values A, B are given instead by Eq. (197). Abbreviate $R_r(y) := R_{a_r, K_r}(y)$. For each index j below, use $10^{-12}Q_j(y)$ as the right-hand side of Eq. (194). Its displayed coefficients specify $\lambda_0, \lambda_1, c_2, c_3, c_6$ and the nonzero d_r ; all other multipliers are zero.

$$\begin{aligned}
Q_1(y) &:= 910063379265 + 138456622547y + 137178141692p_2(y) \\
&+ 7426422730p_6(y) + 78481871075R_0(y).
\end{aligned} \tag{267}$$

$$\begin{aligned}
Q_2(y) &:= 973787270626 + 147514825698y + 137752449007p_2(y) \\
&+ 7292244104p_6(y) + 399496589623R_1(y).
\end{aligned} \tag{268}$$

$$\begin{aligned}
Q_3(y) &:= 987346499573 + 154121153665y + 139437311229p_2(y) \\
&+ 3392914621p_3(y) + 6004014310p_6(y) \\
&+ 406474578592R_2(y).
\end{aligned} \tag{269}$$

$$\begin{aligned}
Q_4(y) &:= 1043197074278 + 192324182126y + 135740668558p_2(y) \\
&+ 13869260900p_3(y) + 3962309142p_6(y) \\
&+ 676219300998R_3(y).
\end{aligned} \tag{270}$$

$$\begin{aligned}
Q_5(y) &:= 1011019607692 + 164080400276y + 188162307589p_2(y) \\
&+ 12126860549p_6(y) + 12652222125R_3(y) \\
&+ 448838667314R_4(y).
\end{aligned} \tag{271}$$

$$\begin{aligned}
Q_6(y) &:= 948621263546 + 145033462238y + 156146426386p_2(y) \\
&+ 8692567530p_6(y) + 201790494821R_3(y).
\end{aligned} \tag{272}$$

$$\begin{aligned}
Q_7(y) := & 990310387361 + 146249407516y + 148853152369p_2(y) \\
& + 675029655p_3(y) + 7037335074p_6(y) \\
& + 12520561544R_5(y) + 377434176095R_6(y).
\end{aligned} \tag{273}$$

$$\begin{aligned}
Q_8(y) := & 970548903165 + 149962106504y + 167246304984p_2(y) \\
& + 9623719261p_6(y) + 279879502285R_7(y).
\end{aligned} \tag{274}$$

$$\begin{aligned}
Q_9(y) := & 949271491460 + 146946715539y + 165592198900p_2(y) \\
& + 9582031208p_6(y) + 181735964211R_8(y).
\end{aligned} \tag{275}$$

$$\begin{aligned}
Q_{10}(y) := & 962884414892 + 145523944844y + 160799013622p_2(y) \\
& + 8753835016p_6(y) + 245804927076R_9(y).
\end{aligned} \tag{276}$$

$$\begin{aligned}
Q_{11}(y) := & 955965651797 + 145931937697y + 163683513580p_2(y) \\
& + 9164193413p_6(y) + 209805889542R_{10}(y).
\end{aligned} \tag{277}$$

$$\begin{aligned}
Q_{12}(y) := & 979427574311 + 147705859673y + 164153625571p_2(y) \\
& + 8905780616p_6(y) + 314065371882R_{11}(y).
\end{aligned} \tag{278}$$

$$\begin{aligned}
Q_{13}(y) := & 1001237674944 + 152152112212y + 172831694569p_2(y) \\
& + 9606070925p_6(y) + 399908803472R_{12}(y).
\end{aligned} \tag{279}$$

$$\begin{aligned}
Q_{14}(y) := & 1005335421380 + 163378978307y + 195830328800p_2(y) \\
& + 12688890699p_6(y) + 17444175054R_{12}(y) \\
& + 390670313664R_{13}(y).
\end{aligned} \tag{280}$$

$$\begin{aligned}
Q_{15}(y) := & 964491337575 + 150976983551y + 174209555026p_2(y) \\
& + 10375432866p_6(y) + 237620185298R_{12}(y).
\end{aligned} \tag{281}$$

$$\begin{aligned}
Q_{16}(y) := & 995710838040 + 152799698011y + 174188953320p_2(y) \\
& + 9915979563p_6(y) + 230173642R_{14}(y) \\
& + 375188945011R_{15}(y).
\end{aligned} \tag{282}$$

$$\begin{aligned}
Q_{17}(y) := & 1011598597281 + 158296238985y + 183060975206p_2(y) \\
& + 10898151455p_6(y) + 16073964625R_{14}(y) \\
& + 249205486483R_{15}(y) + 178134238604R_{16}(y).
\end{aligned} \tag{283}$$

$$\begin{aligned}
Q_{18}(y) := & 963539519127 + 150028413973y + 172779502756p_2(y) \\
& + 10171878204p_6(y) + 233155841929R_{15}(y).
\end{aligned} \tag{284}$$

$$\begin{aligned}
Q_{19}(y) := & 993817646645 + 152251924195y + 173448549513p_2(y) \\
& + 9831045108p_6(y) + 165819017R_{17}(y) \\
& + 367490419037R_{18}(y).
\end{aligned} \tag{285}$$

$$\begin{aligned}
Q_{u_1}(y) := & 952226601904 + 144801742949y + 162184235736p_2(y) \\
& + 9008537093p_6(y) + 190464014501R_{11}(y).
\end{aligned} \tag{286}$$

$$Q_{u_2}(y) := 953559058203 + 147795534038y + 169787815106p_2(y) + 9891614813p_6(y) + 189755337090R_{18}(y). \quad (287)$$

These data determine all rational functions in the prefix. Evaluate entries $0, \dots, 11$, then the first schedule, entries $12, \dots, 19$, and the second schedule. Every reference is available at the step where it is used. Clear the positive denominators in Eq. (196) and in the two compression bounds, and apply the Bernstein procedure in Section 4.12. Bisect at exact midpoints, testing both children, until all terminal coefficients are positive. For a rectangle alternate the two coordinates, beginning with x ; if the polynomial is univariate, split only its nonconstant coordinate. The required depth is at most fourteen. This gives the 76176 positive coefficients reported there. The six additional constraints and the final schedule are printed in Section 4.13.

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